Borel-Weil Type Theorem for the Flag Manifold of a Generalized Kac-Moody Algebra

By Kiyokazu SUTO

Department of Applied Mathematics, Okayama University of Science

(Communicated by Kiyosi ITÔ, M. J. A., May 12, 1995)

In this article, we construct and analyze a certain manifold $\boldsymbol{X}$ associated with a generalized Kac-Moody (GKM) algebra. In the case of a Kac-Moody algebra, $\boldsymbol{X}$ equals the flag manifold constructed in [3]. So we call this manifold the flag manifold of the GKM algebra.

We also give certain kinds of line bundles on the flag manifold $\boldsymbol{X}$, and determine the spaces of global sections of the bundles. The author emphasizes that the analysis of such spaces plays an important role in the highest weight representation theory in the case of a Kac-Moody algebra, especially in the proof of the Kazhdan-Lusztig conjecture.

§1. A generalized Kac-Moody algebra. Let n be a positive integer, I an n-elements set, and $A = (a_{ij})_{i,j\in I}$ a real matrix indexed by I satisfying (1) $a_{ii} = 2$ or ≤ 0, (2) $a_{ij} \leq 0$ if $i \neq j$, and in addition $a_{ij} \in \boldsymbol{Z}$ if $a_{ii} = 2$, and (3) $a_{ij} = 0 \Leftrightarrow a_{ji} = 0$. In this paper, we further assume that A is symmetrizable.

Let $(\mathfrak{h}_{\boldsymbol{R}}, \Pi, \check{\Pi})$ be a realization of A over $\boldsymbol{R}$. We denote by $\mathfrak{g}_{\boldsymbol{R}} = \mathfrak{g}_{\boldsymbol{R}}(A)$ the GKM algebra over $\boldsymbol{R}$ constructed from this realization and the Chevalley generators $e_i, f_i (i \in I)$. Then $\mathfrak{g} = \mathfrak{g}(A) \overset{\text{def}}{=} \boldsymbol{C} \otimes_{\boldsymbol{R}} \mathfrak{g}_{\boldsymbol{R}}$ and $\mathfrak{h} = \boldsymbol{C} \otimes_{\boldsymbol{R}} \mathfrak{h}_{\boldsymbol{R}}$ are the complex GKM algebra with Cartan matrix A and its Cartan subalgebra, respectively. We use the following standard notations:

- $Q_+ = \sum_{i\in I} \boldsymbol{Z}_{\geq 0}\alpha_i$, and $\lambda \geq \lambda' \overset{\text{def}}{\Leftrightarrow} \lambda - \lambda' \in Q_+$,
- $\mathfrak{g}_\alpha = \{x \in \mathfrak{g} \,;\, [h, x] = \alpha(h)x$ for all $h \in \mathfrak{h}\}$ is the root space for $\alpha \in \mathfrak{h}^*$,
- $\Delta = \{\alpha \in \mathfrak{h}^* \setminus \{0\} \,;\, \mathfrak{g}_\alpha \neq 0\}$ is the set of roots,
- $\Delta_+ = \Delta \cap Q_+$ is the set of positive roots, and $\mathfrak{n}_\pm = \sum_{\alpha\in\Delta_+} \mathfrak{g}_{\pm\alpha}$,
- $I^{re} = \{i \in I \,;\, a_{ii} = 2\}$,
- $\Pi^{re} = \{\alpha_i \,;\, i \in I^{re}\}$ is the set of real simple roots,
- $\check{\Pi}^{re} = \{\check{\alpha}_i \,;\, i \in I^{re}\}$ is the set of real simple coroots,
- $r_i : \mathfrak{h}^* \ni \lambda \mapsto \lambda - \lambda(\check{\alpha}_i)\alpha_i \in \mathfrak{h}^* (i \in I^{re})$ are the simple reflections,
- $W = \langle r_i \,;\, i \in I^{re}\rangle \subset \mathrm{GL}(\mathfrak{h}^*)$ is the Weyl group,
- $P = \{\lambda \in \mathfrak{h}^*_{\boldsymbol{R}} \,;\, \lambda(\check{\alpha}_i) \in \boldsymbol{Z}$ for all $i \in I^{re}\}$ is the set of integral weights,
- $P_+ = \{\lambda \in P \,;\, \lambda(\check{\alpha}_i) \geq 0$ for all $i \in I\}$ is the set of dominant integral weights,
- $P_+(I^{re}) = \{\lambda \in P \,;\, \lambda(\check{\alpha}_i) \geq 0$ for all $i \in I^{re}\}$,
- $P_{++} = \{\lambda \in P \,;\, \lambda(\check{\alpha}_i) > 0$ for all $i \in I\}$ is the set of regular dominant integral weights.

We denote by $L(\lambda)$ the irreducible $\mathfrak{g}$-module with highest weight $\lambda \in \mathfrak{h}^*$.

Let V be a $\mathfrak{g}$-module. If V is expressed as the direct sum $\sum_{\mu\in\mathfrak{h}^*} V_\mu$ of the subspaces $V_\mu \overset{\text{def}}{=} \{v \in V \,;\, hv = \mu(h)v$ for all $h \in \mathfrak{h}\}$, then we call the module V $\mathfrak{h}$-diagonalizable, each subspace V_μ the weight space of weight μ, and each element of the set $P(V) \overset{\text{def}}{=} \{\mu \in \mathfrak{h}^* \,;\, V_\mu \neq 0\}$ a weight of V. If in addition e_i and f_i $(i \in I^{re})$ act locally nilpotently, V is called integrable.

Let V be an $\mathfrak{h}$-diagonalizable $\mathfrak{g}$-module with finite-dimensional weight spaces. We denote by V^* the $\mathfrak{g}$-invariant subspace $\sum_{\mu\in P(V)} (V_\mu)^*$ of the $\mathfrak{g}$-module $\mathrm{Hom}_{\boldsymbol{C}}(V, \boldsymbol{C})$. We have $(V^*)_\mu = (V_{-\mu})^*$ for any $\mu \in \mathfrak{h}^*$.

§2. Completions of algebras and modules. Let $w \in W$, and put

$$\Delta_+(w) = \Delta_+ \cap w\Delta_+,\ \mathfrak{n}_\pm(w) = \sum_{\alpha\in\Delta_+(w)} \mathfrak{g}_{\pm\alpha},$$
$$\hat{\mathfrak{n}}_\pm(w) = \prod_{\alpha\in\Delta_+(w)} \mathfrak{g}_{\pm\alpha},\ {}^w\mathfrak{n}_\pm = \sum_{\alpha\in w\Delta_+} \mathfrak{g}_{\pm\alpha},$$
$${}^w\hat{\mathfrak{n}}_\pm = \prod_{\alpha\in w\Delta_+} \mathfrak{g}_{\pm\alpha},\ \hat{\mathfrak{g}} = {}^w\hat{\mathfrak{n}}_- + \mathfrak{h} + {}^w\mathfrak{n}_+.$$

Clearly, ${}^w\mathfrak{n}_\pm$, $\mathfrak{n}_\pm(w)$, and ${}^w\mathfrak{n}_\pm$ are subalgebras of $\mathfrak{g}$. Furthermore, the bracket products in ${}^w\mathfrak{n}_\pm$ (resp. $\mathfrak{g}$) are naturally extended to ${}^w\hat{\mathfrak{n}}_\pm$ (resp. $\hat{\mathfrak{g}}$), and the definition of the Lie algebra $\hat{\mathfrak{g}}$ is independent of w.

Let V be an integrable $\mathfrak{g}$-module such that $P(V) \subset (\lambda_1 - Q_+) \cup \ldots \cup (\lambda_m - Q_+)$ for some $\lambda_1, \ldots, \lambda_m \in \mathfrak{h}^*$. We consider the direct product of weight spaces of V :