Operational Calculus in Several Variables

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1. Introduction. The Theory of Mikusiński's operational calculus is based on the Theorem of Titchmarsh which guarantees that the ring of the complexed-valued continuous functions with convolution as product is an integral domain. He treated a partial differential equation of two variables with constant coefficients as an ordinary differential equation with coefficients in the convolution ring of one variable. But in order to treat equations more symmetrically with respect to several variables, we consider convolution product of functions of several variables. Then, if the differential operator of a partial differential equation with constant coefficients can be factored into linear operators, we are able to treat them just as in the case of one variable. To develope the theory in several variable case, natural definition of convolution of two functions $f(t_1, \cdots, t_m)$ and $g(t_1, \cdots, t_m)$ is

$$(f * g)(t_1, \cdots, t_m) = \int_0^{t_1} \cdots \int_0^{t_m} f(t_1 - u_1, \cdots, t_m - u_m) g(u_1, \cdots, u_m) du_1 \cdots du_m,$$

since we can easily verify that the ring C of m variables on $[0, \infty)^m$ is a commutative ring with this product. It is known ([1]) that this ring is an integral domain. We define partial integration operators and linear partial differential operators based on this theorem and as applications we consider linear differential equations with constant coefficients.

2. The convolution quotients.

Theorem 1 (Generalized Titchmarsh's theorem). *Let A be the convolution ring of continuous functions on R^m which have support in $[0, \infty)^m$. Then A has no divisors of zero.*

The proof is in [1] *Chapter VI.*

To simplify notation we shall consider two variable case. Therefore we denote by C the convolution ring of complex-valued continuous functions $z(x, y)$ with support in $[0, \infty)^2$ which are continuously differentiable on $(0, \infty)^2$ with continuously differentiable $z(x, 0)$ and $z(0, y)$. We assume that all functions in this paper are in C and $\alpha > 0$ and c a complex number. For $f(x, y)$, $g(x, y)$ in C, let $f * g$ be the convolution of f and g with respect to x, y, that is,

$$(f * g)(x, y) = \int_0^x du \int_0^y f(x-u, y-v) g(u, v) dv.$$

Then we can easily verify

$$f * g = g * f,$$
$$f * (g * h) = (f * g) * h.$$

With the usual sum $f(t) + g(t)$, it is clear that

$$h * (f + g) = h * f + h * g.$$

We often omit '$*$' if it is clear from the context. To stress that we consider a function $f(x, y)$ as an element of the ring C, we denote it by $\{f(x, y)\}$. By the generalized Titchmarsh's theorem, the ring C is without zero-divisor. Let Q be the field of total fractions of C and we shall call each element of Q an inner operator and any other operator operating on C or Q an outer operator. Functions in C may not necessarily be continuous on x and y axes, but we have

Proposition 2. *The ring C has no divisors of zero.*

Proof. Let f, g be in C and $f * g = 0$. Then $1 * f$ and $1 * g$ are in A. Since $(1 * f) * (1 * g) = 0$, either $1 * f = 0$ or $1 * g = 0$. Differentiating $1 * f = 0$ or $1 * g = 0$ by x and y, we have either $f(x, y) = 0$ or $g(x, y) = 0$ for (x, y) in $(0, \infty)^2$. Since f, g are continuous on $[0, \infty)^2$ and with support in $[0, \infty)^2$, we have $f(x, y) = 0$ or $g(x, y) = 0$ identically.

Definition. We define the unit operator I by $I = \dfrac{\{1\}}{\{1\}}$.

We shall denote by h the operator defined by the function $\{1\}$:

$$h = \{1\}.$$

Then, for any natural number n, we have, by induction on n,

$$h^n = \left\{\frac{x^{n-1} y^{n-1}}{(n-1)!(n-1)!}\right\}.$$