On the Difinition of the Virtanen Property for Riemannian Manifolds

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In our former paper [4] we introduced a notion which we called the *Virtanen property* for Riemannian manifolds. The property is always fulfilled by two dimensional Riemannian manifolds so that it often ensures the possibility of extending certain potential theoretic results valid for two dimensional case to higher dimensions. The purpose of this paper is to give a new definition of the Virtanen property which is equivalent to but more understandable than that given in [4].

Throughout this paper we let M be a noncompact, connected and orientable Riemannian manifold of class C^∞ of dimension $n \geqq 2$. Let (g_{ij}) be the metric tensor on M and $(g^{ij}) = (g_{ij})^{-1}$. With an s-form α on M $(0 \leqq s \leqq n)$ whose local expression in a local parameter $x = (x^1, \ldots, x^n)$ is

$$\alpha = \sum_{i_1 < \cdots < i_s} a_{i_1 \cdots i_s}(x) dx^{i_1} \wedge \cdots \wedge dx^{i_s}$$

we associate a nonnegative function $|\alpha|$ on M, usually referred to as the *point norm* of α, given by

$$(1) \quad |\alpha|^2 = \sum_{i_1 < \cdots < i_s} \Big(\sum_{j_1, \cdots, j_s} g^{i_1 j_1} \cdots g^{i_s j_s} a_{j_1 \cdots j_s} \Big) a_{i_1 \cdots i_s}.$$

If α is measurable, then we can consider its p-norm $(1 \leqq p \leqq \infty)$

$$\| \alpha \|_p = \left(\int_M |\alpha|^p dV \right)^{1/p} \ (1 \leqq p < \infty),$$

$$\| \alpha \|_\infty = \operatorname*{ess.\,sup}_M |\alpha|,$$

where dV is the volume element on M. Using these notations we can give our new definition of the Virtanen property:

Definition. *The manifold M is said to possess the Virtanen property if for any C^∞ $(n-2)$-form α on M with $\| d\alpha \|_2 < \infty$ there exists a sequence (α_m) of C^∞ $(n-2)$-forms α_m on M such that*

$$(2) \quad \| \alpha_m \|_\infty < \infty \ (m = 1, 2, \ldots),$$

$$(3) \quad \lim_{m \to \infty} \| d\alpha - d\alpha_m \|_2 = 0.$$

If the dimension $n = 2$, then the given form α and sought forms α_m are 0-forms, i.e. functions, on M. Taking α_m as a suitable regularization of the function $\max(\min(\alpha, m), -m)$ for each $m = 1, 2, \ldots$, we see that (2) and (3) are satisfied by these α and α_m (cf. e.g. [3]) so that the Virtanen property is always possessed by any two dimensional Riemannian manifold M. In our former definition of the Virtanen property in [4] we had

$$(4) \quad N[\alpha_m] := \sup\{ \| \alpha_m \wedge d\varphi \|_2 / \| d\varphi \|_2 : \varphi \in C_0^\infty(M) \setminus \{0\} \} < \infty$$

instead of (2). The function norm $\| \alpha \|_\infty$ is much easier to compute than the operator norm $N[\alpha]$ so that we may say that our new definition is better than our former one. To assure that these two definitions are actually equivalent we have to prove that these two norms are equivalent. The practical purpose of this paper is, thus, to prove the following

Theorem. *The norms $\| \alpha \|_\infty$ and $N[\alpha]$ for any C^∞ $(n-2)$-form α are equivalent, i.e. the following inequalities are valid for every C^∞ $(n-2)$-form α on M:*

$$(5) \quad (n/2)^{-1/2} \| \alpha \|_\infty \leqq N[\alpha] \leqq \| \alpha \|_\infty.$$

Observe that (5) implies $N[\alpha] = \| \alpha \|_\infty$ for $n = 2$, which we already remarked in [4]. Inequalities in (5) are *sharp* in the following sense: for every dimension $n \geqq 2$, there is a couple (M, α) such that $N[\alpha] = (n/2)^{-1/2} \| \alpha \|_\infty > 0$ and also $N[\alpha] = \| \alpha \|_\infty > 0$. Such examples will be given right after the proof of (5).

Proof of Theorem. A parametric neighborhood $(U; x)$ at $\xi \in M$ is always supposed to satisfy $x(\xi) = 0$. We say that a parametric neighborhood $(U; x)$ at $\xi \in M$ is *special* if the components of the metric tensor $(g_{ij}(x))$ in the local parameter x takes the form $g_{ij}(0) = \delta_{ij}$ so that $g^{ij}(0) = \delta^{ij}$ as well $(i, j = 1, \ldots, n)$.

We start with the proof for the second inequality in (5) which is simple. Take any $\varphi \in$

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