A Mean Value Property in Adele Geometry

By Masanori MORISHITA

Department of Mathematics, Kanazawa University

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Introduction. Let X be a left homogeneous space of a connected linear algebraic group G. Suppose that G, X and the action are defined over $\boldsymbol{Q}$, the field of rational numbers, and that X has a $\boldsymbol{Q}$-rational point x. We then identify X with G/H, where H is the stabilizer of x.

After the works of Siegel [13] and Weil [14], Ono [10] investigated a mean value theorem for the adele space attached to a *uniform* and *special* homogeneous space $X = G/H$, introducing the Tamagawa number $\tau(G, X)$. Here, X is said to be *special* if G and H are connected linear $\boldsymbol{Q}$-groups without tori parts in their Levi-Chevalley decompositions.

In [8], using Kottwitz's fundamental theorem on the Tamagawa number [6], we showed that any special homogeneous space is uniform, and gave a formula expressing $\tau(G, X)$ in terms of the fundamental groups of G and H.

The purpose of this paper is to give a generalization of our results for special homogeneous spaces to those for a wider class of homogeneous spaces allowing G and H to have $\boldsymbol{Q}$-anisotropic tori in their Levi-Chevalley decompositions. Since a reductive group does not have a universal covering in general, we use Borovoi's algebraic fundamental group to describe our results. Also, we use his theory on abelian Galois cohomology which is a machinery to study Galois cohomology of connected linear algebraic groups in a functorial way ([1], [2], [3] and Appendix B to [7]).

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1. Borovoi's fundamental group and abelian Galois cohomology. In this section, we introduce Borovoi's algebraic fundamental group and abelian Galois cohomology which we need later to describe our results. For these matters, we refer to [1], [2], [3], and also Appendix B to [7].

Let k be a field of characteristic zero and $\bar{k}$ a fixed algebraic closure of k. First, we assume that G is reductive. Let G^{ss} be the derived group of G and G^{sc} be the universal k-covering of G^{ss} [9], Appendix I). Consider the composition

$$\rho : G^{sc} \to G^{ss} \subset G.$$

Take a maximal torus T in $G_{\bar{k}}$ and put $T^{sc} = \rho^{-1}(T)$. We then define

$$\pi_1(G, T) := X_*(T)/\rho_* X_*(T^{sc}),$$

where $X_*(S)$ denotes the group of one-parameter subgroups of a torus S. If T' is another maximal torus in $G_{\bar{k}}$, there is $g \in G(\bar{k})$ so that $T' = gTg^{-1} = Int(g)(T)$. Then, $Int(g)$ induces the isomorphism $g_* : \pi_1(G, T) \simeq \pi_1(G, T')$ which does not depend on the choice of g. The Galois group $\mathrm{Gal}(\bar{k}/k)$ acts on $\pi_1(G, T)$ in the following way. For $\sigma \in \mathrm{Gal}(\bar{k}/k)$, there is $g_\sigma \varepsilon G(\bar{k})$ so that $T^\sigma = g_\sigma^{-1} T g_\sigma$. Then, σ acts on $\pi_1(G, T)$ as the composition

$$\pi_1(G, T) \overset{\sigma_*}{\to} \pi_1(G, T^\sigma) \overset{(g\sigma)_*}{\to} \pi_1(G, T).$$

We see that the above isomorphism g_* is $\mathrm{Gal}(\bar{k}/k)$-equivariant. So, we simply write $\pi_1(G)$ for this Galois module. For a connected linear k-group G, we set $\pi_1(G) := \pi_1(G/G^u)$, where G^u is the unipotent radical of G, and call it *Borovoi's fundamental group* of G. Then, $\pi_1(\cdot)$ is an exact functor from the category of connected linear k-groups to $\mathrm{Gal}(\bar{k}/k)$-modules, finitely generated over $\boldsymbol{Z}$. One sees that an inner twisting $G \to G'$ induces the isomorphism $\pi_1(G) \simeq \pi_1(G')$, and that if $k \subset \boldsymbol{C}$, $\pi_1(G)$ is canonically isomorphic to the topological fundamental group of the complex Lie group $G(\boldsymbol{C})$ as abelian groups.

Next, we define the abelian Galois cohomology groups of a connected reductive group G by

$$H^i_{ab}(k, G) := \boldsymbol{H}^i(k, T^{sc} \to T) \ (i \geq -1),$$

where $\boldsymbol{H}^i$ means the Galois hypercohomology of the complex

$$0 \to T^{sc} \to T \to 0,$$

where T^{sc} and T sit in degree -1 and 0, respectively.