A Remark on Integral Representations Associated with p-adic Field Extensions

By Shuji YAMAGATA

Department of Mathematical Sciences, Tokyo Denki University

(Communicated by Shokichi IYANAGA, M. J. A., Nov. 13, 1995)

Let K be a local field of characteristic 0 with algebraically closed residue field of characteristic $p > 0$. In this paper, an extension of K means an extension of K contained in some fixed algebraic closure $\bar{K}$ of K. Let K_∞/K be a $\boldsymbol{Z}_p$-extension with Galois group $\Gamma = \mathrm{Gal}(K_\infty/K)$ ($\cong \boldsymbol{Z}_p$). Let $\Gamma_n = \Gamma^{p^n}$ and K_n the subfield of K_∞ fixed by Γ_n. Denote by $\mathcal{O}(F)$ the ring of integers of an extension F/K. Especially put $\mathcal{O}_n = \mathcal{O}(K_n)$ and $\mathcal{O} = \mathcal{O}(K)$. For a product R of extensions of K, $\mathcal{O}(R)$ denotes the product of the rings of integers of the factors i.e. the unique maximal order of R. For two finite extensions F/K and F'/K, let F_i, $i = 1,2,\ldots,f$ be all the composite fields of the images of K-embeddings of F into $\bar{K}$ (up to equivalence of proper embeddings of F above F' in the sense of [4])) with F'. Then we have $F \otimes_K F' \cong \Pi F_i$. Put $F_{\otimes m} = F \otimes_K K_m$.

We attach, to any finite extension E/K, the $\mathcal{O}_m$-semi-linear representation $\mathcal{O}(E_{\otimes m})$ of Γ/Γ_m given by its Galois action on K_m. In [3] S. Sen proved (probably in collaboration with J-M. Fontaine): Let E/K and E'/K be two finite Galois p-extensions.E/K and E'/K are isomorphic if and only if, for some sufficiently large m, the $\mathcal{O}_m$-semi-linear representations of Γ/Γ_m on the additive groups $\mathcal{O}(E_{\otimes m})$ and $\mathcal{O}(E'_{\otimes m})$ are isomorphic. In [1], F. Destrempes generalized this theorem for two finite Galois extensions.

The purpose of this paper is to prove the following theorem:

Theorem (cf. Theorem 2 of [3] and Theorem 1 of [1]). Let E/K and E'/K be two finite extensions. Assume that, for some sufficiently large m (cf. Remark 1 of §2), the $\mathcal{O}_m$-semi-linear representations of Γ/Γ_m on the additive groups $\mathcal{O}(E_{\otimes m})$ and $\mathcal{O}(E'_{\otimes m})$ are isomorphic. Then the Galois closures of E/K and E'/K coincide and $\deg E/K = \deg E'/K$.

The author would like to express his hearty thanks to Professor Keiichi Komatsu for his advice and encouragements.

§1. Preliminaries. For a finite extension F/K, let π_F be a prime element of F and v_F the valuation of F normalized by $v_F(\pi_F) = 1$. Especially put $\pi_n = \pi_{K_n}$ and $v_n = v_{K_n}$.

The following proposition is a generalization of Proposition 6 of [3] and Proposition 6 of [1].

Proposition 1. Let E/K and E^*/K be two finite extensions. Then there is an integer s, independent of m, such that

$$\mathcal{O}(E_{\otimes m} \otimes_{K_m} E^*_{\otimes m})/(\mathcal{O}(E_{\otimes m}) \otimes_{\mathcal{O}_m} \mathcal{O}(E^*_{\otimes m}))$$

is killed by π_m^s. Here s depends only on one of the two extensions E/K and E^*/K.

Proof. Let F/K be a finite extension. We claim that, for sufficiently large m, $v_m(\delta(FK_m/K_m))$ has an upper bound which depends only on F/K, not on m. Here $\delta(FK_m/K_m)$ is the discriminant ideal of the extension FK_m/K_m. If F/K is a finite Galois p-extension, the assertion was proved in Lemma 1 of [1]. General case follows from it by considering the Galois closure and using transitivity of discriminant. Hence we have proved the proposition by Lemma 4 of [1].

The next elementary lemma is used in the following.

Lemma. Let E/K be a finite extension and F/K a finite Galois extension. Write $E \otimes_K F \cong \Pi E_i$ as the product of the composite fields. Then $\deg E_i/K$ does not depend on i. Furthermore, if $\deg E/K$ and $\deg F/K$ are powers of p, so is $\deg E_i/K$.

Proof. Write $E \cong K[x]/(f)$ with an irreducible monic polynomial $f \in K[x]$. We have $E \otimes_K F \cong F[x]/(f) \cong \Pi F[x]/(f_i)$ if we decompose f into the product Πf_i of irreducible monic polynomials in $F[x]$ (cf. for example, Lemma 6, Chap. 2, §5.2 of [2]). As F/K is a Galois extension, f_i's are conjugate under $\mathrm{Gal}(F/K)$-action on the coefficients. Thus $\deg E_i/K = \deg f_i$ does not depend on i.