Elliptic Genera and Vertex Operator Super Algebras

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Abstract: The elliptic genus for a closed Riemannian Spin manifold, when regarded as a pair of graded vector spaces, is shown to have the structure of a pair of modules over a vertex operator super algebra of parallel sections of an LSpin bundle. Some interesting parallel sections and the corresponding vertex operators are described for various subclasses of Riemannian manifolds defined by parallel geometric structures. In particular, vertex operators corresponding to Kähler forms generate affine Lie algebras and thus the elliptic genera are their representations.

Key words: Elliptic genera; Spin representations; vertex operator super algebras; Virasoro algebras; affine Lie algebras; Kähler manifolds; modular functions.

1. Introduction. The elliptic genus of a closed Spin manifold M^{2N} is the S^1-equivariant signature of its loop space LM with respect to the canonical S^1-action on LM [7]. In algebraic topology, it has been studied as a modular form valued genus, or a ring map, from cobordism rings [4]. This point of view leads to the construction of so-called elliptic cohomology. The name "elliptic" comes from the fact that the logarithm series associated to this genus can be expressed in terms of an elliptic integral of a Jacobi quartic [5]. The construction of elliptic genera involves graded vector bundles arising from the Spin representation V, W of the orthogonal affine Lie algebra $\hat{\mathfrak{o}}(2N)$, each of which is a sum of two level 1 irreducible representations.

It is known that one of these Spin representations V has the structure of a vertex operator super algebra (VOA) with W as its module [2]. To any vector v in a vertex operator super algebra V, there corresponds a family of infinitely many operators $\{v\}_n$, $n \in (1/2)\boldsymbol{Z}$, acting on the algebra itself and its modules. The main structure of a vertex operator super algebra is that the totality of these operators satisfy a Jacobi identity which is a generalization of the usual Jacobi identity for Lie algebras. Our result gives rise to a geometric construction of various vertex operator super algebras and their modules.

First we will observe that when a closed Spin manifold M has a Riemannian structure, the elliptic genus $\Phi_{\mathrm{ell}}(M)$ has the structure of a pair of graded vector spaces. If we take the graded dimension, we get the modular function valued elliptic genus. The Spin representation V of an orthogonal affine Lie algebra $\hat{\mathfrak{o}}(2N)$ gives rise to a graded vector bundle ν_M of generalized differential forms on M with a covariant derivative induced from the Levi-Civita connection on TM. Let $\mathcal{P}_M = \mathcal{P}(\nu_M)$ be the graded vector space of parallel sections in ν_M.

Theorem 1. *For a compact Riemannian Spin manifold M without boundary, the graded vector space $\mathcal{P}_M$ of generalized parallel differential forms has the structure of a vertex operator super algebra and the elliptic genus $\Phi_{\mathrm{ell}}(M)$ has the structure of a super-pair of modules over $\mathcal{P}_M$. The diagonal action of $\mathcal{P}_M$ on $\Phi_{\mathrm{ell}}(M)$ is effective if the Spin index $\hat{A}(M)$ of M doesn't vanish.*

Thus, the elliptic genus is a geometric device which produces a super-pair of modules $\Phi_{\mathrm{ell}}(M)$ over a vertex operator super algebra $\mathcal{P}_M$ for each closed Riemannian Spin manifold M^{2N}.

$$\left\{\begin{matrix}\text{Closed Riemannian}\\ \text{Spin manifolds}\end{matrix}\right\} \xrightarrow{\text{elliptic genus}} \left\{\begin{matrix}\text{pairs of vertex operator super}\\ \text{algebras and their modules}\end{matrix}\right\}$$
$$M \longrightarrow (\mathcal{P}_M, \Phi_{\mathrm{ell}}(M))$$

The vertex operator super algebra $\mathcal{P}_M$ de-

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