On Cartier-Voros Type Selberg Trace Formula for Congruence Subgroups of $PSL(2, \mathbf{R})$

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1. Introduction. Let Γ be a discrete subgroup of $G = PSL(2, \mathbf{R})$. The group Γ acts on the upper half plane $\mathbf{H}$ by the usual linear fractional transformation. We assume that the fundamental domain of Γ, which is denoted by $\Gamma \backslash \mathbf{H}$, is a finite volume surface with the hyperbolic metric. The Laplacian Δ acting on the space $L^2(\Gamma \backslash \mathbf{H})$ has the spectrum consisting of the discrete and continuous spectra in general. In this setting, as two different expression of the trace of an G-invariant integral operator $L: f(z) \to \int_{\Gamma \backslash \mathbf{H}} \hat{K}(z, z') f(z') dz' (f \in L^2(\Gamma \backslash \mathbf{H}))$ with the kernel function

$$\hat{K}(z, z') = \sum_{\sigma \in \Gamma} k(\sigma z, z') - \hat{H}(z, z'),$$

where k is a point pair invariant and $\hat{H}(z, z')$ is so defined that the continuous spectrum of Δ disappears, Selberg showed his famous trace formula of the following form (see [9]): for any function $h(\rho)$ $(\rho \in \mathbf{C})$, which we call a test function, satisfying the condition A below,

$$D(h) = I(h) + H(h) + E(h) + CP(h),$$

where the left hand side $D(h) = \sum_{n=0}^{\infty} h(\rho_n)$ is the expansion of $Tr(L) = \int_{\Gamma \backslash \mathbf{H}} \hat{K}(z, z) dz$ as the sum of the eigenvalue $h(\rho_n)$ of L corresponding to the discrete spectrum $\lambda_n = \frac{1}{4} + \rho_n^2$ of Δ, i.e. $L\varphi_n = h(\rho_n)\varphi_n$ for $\Delta\varphi_n = \lambda_n \varphi_n$, the right hand side is the expansion of $Tr(L)$ with respect to the conjugacy classes of Γ, and $I(h)$ (resp. $H(h)$, $E(h)$) is the contribution of the identity (resp. hyperbolic, elliptic) conjugacy class of Γ, and $CP(h)$ is the sum of the contribution of the parabolic conjugacy classes of Γ and the contribution of $\hat{H}(z, z')$. This formula has been one of the important objects of study in analytic number theory. Especially the studies of the relations among the Selberg zeta functions $Z_\Gamma(s)$ (see §2) which is induced from the term $H(h)$ of this formula for a special test function, the arithmetic zeta functions, and the spectral zeta functions are very interesting in view of the 'unifying' theory of various zeta functions.

The condition A for a test function $h(\rho)$ is as follows:

(1) $h(-\rho) = h(\rho)$,

(2) $h(\rho)$ is holomorphic in the strip $|\mathrm{Im}\,\rho| < c \left(\text{for some } c > \frac{1}{2}\right)$,

(3) $h(\rho) = O(|\rho|^{-\alpha})$ $(\alpha > 2)$ as $|\rho| \to \infty$ in the above strip.

According to the condition A, for example, $h(t, \rho) = e^{-t\rho} (t > 0)$ can not be taken as a test function.

Now we consider the theta type function $\Theta_\Gamma(t) = \sum_{n=0}^{\infty} e^{-t\rho_n} (t > 0)$ associated with the operator $\sqrt{\Delta - \frac{1}{4}}$ on $L^2(\Gamma \backslash \mathbf{H})$ (see §4). This function $\Theta_\Gamma(t)$ is very interesting because of the following view points: first, $\Theta_\Gamma(t)$ is an analogue of theta functions associated with Δ (see [8]); and secondly, $\Theta_\Gamma(t)$ is also an analogue of that associated with the zeros of zeta functions (see [1], [4], [5]). However, as stated above, the Selberg trace formula can be of no help to study $\Theta_\Gamma(t)$.

But for co-compact discrete subgroups Γ, Cartier-Voros[2] showed the modified Selberg trace formula to study $\Theta_\Gamma(t)$. In this case, Γ has no elliptic and parabolic conjugacy class and no continuous spectrum. This Cartier-Voros type modified formula demands the following condition B for a test function $h(\rho)$.

(1) $h(\rho)$ is holomorphic in an open set $V \subset \mathbf{C}$ containing the closed half plane $\mathrm{Re}\,\rho \geq 0$,

(2) $h(\rho) = O(|\rho|^{-\alpha})$ $(\alpha > 2)$ as $|\rho| \to \infty$ in the strip $|\mathrm{Im}\,\rho| < c$(for some $c > 0$),

(3) $h(\rho) d \log Z_\Gamma\left(\frac{1}{2} + i\rho\right)$ is integrable in $-\frac{\pi}{2} \leq \arg \rho \leq 0$, and $h(\rho) d \log Z_\Gamma\left(\frac{1}{2} - i\rho\right)$ is integrable in $0 \leq \arg \rho \leq \frac{\pi}{2}$.