On Poles of Twisted Tensor L-functions*)

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Abstract: It is shown that the only possible pole of the twisted tensor L-functions in $\mathrm{Re}(s) \geq 1$ is located at $s = 1$ for *all* quadratic extensions of global fields.

0. Introduction. Let E be a quadratic separable field extension of a global field F. Denote by $\mathbf{A}_E$, $\mathbf{A}_F$ the corresponding rings of adeles. Put G_n for GL_n and Z_n for its center. Then $\mathrm{Z}_n(\mathbf{A}_E)$ is the group $\mathbf{A}_E^\times$ of ideles of $\mathbf{A}_E$. Fix a cuspidal representation π of the adele group $\mathrm{G}_n(\mathbf{A}_E)$. Without lost of generality, we may assume that the central character of π is trivial on the split component of $\mathbf{A}_E^\times$. This is the multiplicative group $\mathbf{R}^\times$ of the field of real numbers embedded in $\mathbf{A}_E^\times$ via $x \mapsto (x, \ldots, x, 1, \ldots)$ (x in the archimedean, 1 in the finite components). Let S be a finite set of places of F (depending on π), including the places where E/F ramify, and the archimedean places, such that for each place v' of E above a place v outside S the component $\pi_{v'}$ of π is unramified. Following [1], let r be the twisted tensor representation of $\hat{\mathrm{G}} = [\mathrm{GL}(n, \mathbf{C}) \times \mathrm{GL}(n, \mathbf{C})] \times \mathrm{Gal}(E/F)$ on $\mathbf{C}^n \otimes \mathbf{C}^n$. It acts by $r((a, b))(x \otimes y) = ax \otimes by$ and $r(\sigma)(x \otimes y) = y \otimes x$ ($\sigma \in \mathrm{Gal}(E/F)$, $\sigma \neq 1$). Let q_v be the cardinality of the residue field $R_v/\pi_v R_v$ of the ring R_v of integers in F_v. We define the twisted tensor L-function to be the Euler product

$$L(s, r(\pi), S) = \prod_{v \notin S} \det\,[1 - q_v^{-s} r(t_v)]^{-1}.$$

The representation π is called *distinguished* if its central character is trivial on $\mathbf{A}_F^\times$ and there is an automorphic form $\phi \in \pi$ in $L^2(\mathrm{G}_n(E)\mathrm{Z}_n(\mathbf{A}_F)\backslash \mathrm{G}_n(\mathbf{A}_E))$, such that $\int \phi(g)dg \neq 0$. The integral is taken over the closed subspace $\mathrm{G}_n(F)\mathrm{Z}_n(\mathbf{A}_F)\backslash \mathrm{G}_n(\mathbf{A}_F)$ of $\mathrm{G}_n(E)\mathrm{Z}_n(\mathbf{A}_F)\backslash \mathrm{G}_n(\mathbf{A}_E)$.

The following theorem is proven in [1, p. 309] for a quadratic extension E/F of global fields, such that each archimedean place of F splits in E. We prove it for any quadratic extension of global fields, i.e. also for number fields with completions $E_v/F_v = \mathbf{C}/\mathbf{R}$.

Theorem. *The product $L(s, r(\pi), S)$ converges absolutely, uniformly in compact subsets, in some right half-plane. It has analytic continuation as a meromorphic function to the right half plane $\mathrm{Re}(s) > 1 - \varepsilon$, for some small $\varepsilon > 0$. The only possible pole of $L(s, r(\pi), S)$ in $\mathrm{Re}(s) > 1 - \varepsilon$ is simple, located at $s = 1$. The function $L(s, r(\pi), S)$ has a pole at $s = 1$ if and only if π is distinguished.*

Proof. The proof of this theorem is the same as that of the Theorem of [1, §4], pp. 309-310. On lines 14 and 18 of page 310 of [1], we use the proposition below. It holds in the non-split archimedean case too. Hence the restriction put in [1] on the extension E/F can be removed.

For the functional equation satisfied by $L(s, r(\pi), S)$, see [1]. For the local L-factors at all non-archimedean places of F, see [2]. The non-vanishing of this L-function on the edge $\mathrm{Re}(s) = 1$ of the critical strip has been shown by Shahidi [6]. Twisted tensor L-functions are used in the study (see Kon-no [5]) of the residual spectrum of unitary groups.

1. Local computations. From now on, we consider the local case only. Let E/F be a quadratic extension of local fields. Thus in the archimedean case $E/F = \mathbf{C}/\mathbf{R}$. Denote by $x \mapsto \bar{x}$ the non-trivial automorphism of E over F. Let $\iota \neq 0$ be an element of E, such that $\bar{\iota} = -\iota$. Put G_n for GL_n. The groups of F and E-points are denoted by $\mathrm{G}_n(F)$ and $\mathrm{G}_n(E)$. Denote by N_n the unipotent radical of the upper triangular subgroup of G_n, and by A_n the diagonal subgroup. Let ψ_0 be a non trivial additive character of F. For example, if $F = \mathbf{R}$ then $\psi_0(x) = e^{2\pi i x}$. Let ψ

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