Triangles and Elliptic Curves. IV

By Takashi ONO

Department of Mathematics, The Johns Hopkins University, U.S.A.

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This is a continuation of my preceding papers [1], [2], [3], which will be referred to as (I), (II), (III) in this paper. As in (II), (III), to each triple (l, m, n) of independent linear forms on $\bar{k}^3$, k being a field of characteristic not 2 and $\bar{k}$ its algebraic closure, we associate a space

(0.1) $T = \{t \in \bar{k}^3 ;$
$(l^2 - m^2)(m^2 - n^2)(n^2 - l^2) \neq 0\}.$

Since the condition for $t \in T$ in (0.1) is given by a homogeneous polynomial, we can speak of the subset $P(T)$ of the projective plane

(0.2) $P(T) = \{[t] \in P^2(\bar{k}) ;$
$(l^2 - m^2)(m^2 - n^2)(n^2 - l^2) \neq 0\},$

which is the complement of the complete quadrangle given by six lines $(l^2 - m^2)(m^2 - n^2)(n^2 - l^2) = 0$. Since T is the total space of a bundle whose fibres are (affine parts of) elliptic curves in $P^3(\bar{k})$, it is natural to think of their images under the canonical map $T \to P(T)$ given by $t \mapsto [t]$, the homogeneous coordinates for t. In this paper, we shall study this aspect of the space T and show that there is a close relation between certain family of elliptic curves and a single plane conic, over a given field k of rationality. If X denotes a set of geometric objects, we shall denote by $X(K)$ (or by X_K occasionally) the subset of X which is rational over K.

§1. Basic diagram. Along with the canonical map $P : T \to P(T)'((0,1), (0,2))$, we consider the diagram:

$$(1.1) \qquad \begin{array}{ccc} T & \xrightarrow{P} & P(T) \\ p \downarrow & & \downarrow \bar{p} \\ \Omega & \xrightarrow{r} & \Lambda \end{array}$$

where

(1.2) $\Omega = \{\omega = (M, N) \in \bar{k} \times \bar{k} ;$
$MN(M - N) \neq 0\},$

(1.3) $\Lambda = \{\lambda \in \bar{k} ; \lambda \neq 0,1\},$

(1.4) $p(t) = (l^2 - n^2, m^2 - n^2), \; r(\omega) = \dfrac{N}{M},$

(1.5) $\bar{p}[t] = r(p(t)) = \dfrac{m^2 - n^2}{l^2 - n^2}.$

Since $\bar{k}$ is algebraically closed, p is surjective and hence so is $\bar{p}$. For an $\omega = (M, N) \in \Omega$, P induces naturally a map

(1.6) $P_\omega : p^{-1}(\omega) \to \bar{p}^{-1}(r(\omega)).$

Again since $\bar{k}$ is algebraically closed, we see that P_ω is surjective and each fibre is of the form $\{\pm t\}$, $t \in T$; in other words, P_ω is a covering of degree 2. The fibres of $p, \bar{p}$ are described as follows. For an $\omega = (M, N)$, let

(1.7) $E(\omega) = \{[x] \in P^3(\bar{k}) ;$
$x_0^2 + Mx_1^2 = x_2^2, \; x_0^2 + Nx_1^2 = x_3^2\},$

this being an elliptic curve in $P^3(\bar{k})$ (see e.g., [4] Chap. 4). Deleting four 2-torsion points out of (1.7), we obtain the affine part of (1.7):

(1.8) $E_0(\omega) = \{(x, y, z) \in \bar{k}^3 ; z^2 + M = x^2,$
$z^2 + N = y^2\}.$

From (1.4), (1.8), we have a bijection

(1.9) $p^{-1}(\omega) \xrightarrow{\sim} E_0(\omega), \; \omega \in \Omega,$

given by $t \mapsto (l(t), m(t), n(t))$, $t \in p^{-1}(\omega)$.

On the other hand, for a $\lambda \in \Lambda$, let

(1.10) $c(\lambda) = \{[x, y, z] \in p^2(\bar{k}) ;$
$y^2 - z^2 = \lambda(x^2 - z^2)\},$

this being a nonsingular conic in $p^2(\bar{k})$. Denoting by H the complete quadrangle given by

(1.11) $H = \{[x, y, z] \in p^2(\bar{k}) ;$
$(x^2 - y^2)(y^2 - z^2)(z^2 - x^2) = 0\},$

we have

(1.12) $C(\lambda) \cap H = \{[1,1,1], [-1,1,1],$
$[1, -1,1], [1,1, -1]\}$

which is independent of $\lambda \in \Lambda$.

Deleting these four points from $C(\lambda)$, write

(1.13) $C_0(\lambda) = C(\lambda) - H.$

From (1.5), (1.11), (1.12), (1.13), we have a bijection

(1.14) $\bar{p}^{-1}(\lambda) \xrightarrow{\sim} C_0(\lambda)$

given by $[t] \mapsto [l(t), m(t), n(t)]$.

In view of (1.6), (1.9), (1.14), we obtain a covering of degree 2:

(1.15) $\pi_\omega : E_0(\omega) \to C_0\left(\dfrac{N}{M}\right), \; \omega = (M, N) \in \Omega,$

given by $(x, y, z) \mapsto [x, y, z]$.