20. *Some Estimates for Eigenvalues of Schrödinger Operators*

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1. Introduction. In this paper we give estimates for large eigenvalues of Schrödinger operators $-\Delta + V$ with increasing potential V. Let $N(\lambda)$ be the number of eigenvalues of the Schrödinger operator less than λ. Under some conditions on V we can prove the asymptotic formula

(1) $N(\lambda) \sim (2\pi)^{-d} \mid \{(\xi, x) \in \mathbf{R}^d \times \mathbf{R}^d : |\xi|^2 + V(x) < \lambda\} \mid (\lambda \to \infty)$,

which means that there is a correspondence between each eigenvalue less than λ and each set with volume $(2\pi)^d$ in $\{(\xi, x) \in \mathbf{R}^d \times \mathbf{R}^d : |\xi|^2 + V(x) < \lambda\}$. This correspondence is known as the Bohr-Sommerfeld quantization rule. A lot of people study the conditions on potentials for the formula (1), for instance, Feigin [3], Fleckinger [4], Rozenbljum [5], Simon [6], Tachizawa [7], Titchmarsh [8] and so on.

In this paper we give another formulation of this problem. Let $A = (\mathbf{N} \times \mathbf{Z}) \cup \{(0,2n') : n' \in \mathbf{Z}\}$ and $B = \{(m, n) : m = (m_1, \dots, m_d), n = (n_1, \dots, n_d), (m_i, n_i) \in A, i = 1, \dots, d\}$. Our claim is that there is a correspondence between each eigenvalue and each point $(2\pi m, n)$ for $(m, n) \in B$. Let $\theta_{m,n} = |2\pi m|^2 + V(n/2)$ for $(m, n) \in B$ and $\{\mu_k\}_{k\in\mathbf{N}}$ the rearrangement of $\{\theta_{m,n}\}_{(m,n)\in B}$ in the nondecreasing order. We show that

$$\lim_{k\to\infty} \frac{\lambda_k}{\mu_k} = 1 \tag{2}$$

under some conditions on V. The formula (2) gives a relation between the asympototic behavior of eigenvalues and the symbol of the Schrödinger operator, which is a new result.

The class of the potentials V studied in this paper contains slowly increasing ones, for example, $V(x) = \log \cdots \log |x|$ (large $|x|$). The formula (1) is proved in [7] for radial, slowly increasing potentials. But it is not known whether the formula (1) holds or not for non-radial slowly increasing potentials. Our theorem gives a new approach to the study of eigenvalues of Schrödinger operators with slowly increasing potentials.

2. Theorem. We consider potentials $V(x)$ satisfying the following conditions.

(H1) $V \in C^\infty(\mathbf{R}^d)$, $V \geq 1$, $V(x) \to \infty$ ($|x| \to \infty$).

(H2) There are positive constants c, γ such that

$$V(x + y) \leq c(1 + |y|)^\gamma V(x) \quad (x, y \in \mathbf{R}^d).$$

(H3) There is a constant τ, $1/2 \leq \tau < 1$, such that, for every $\alpha = (\alpha_1, \dots, \alpha_d) \in \mathbf{Z}_+^d$, $1 \leq |\alpha| = \alpha_1 + \cdots + \alpha_d$,

$$|\partial_x^\alpha V(x)| \leq C_\alpha V(x)^\tau \quad (x \in \mathbf{R}^d)$$

where C_α is a positive constant depending only on α.