74. *On Jacobi-Perron Algorithm*

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§1. Introduction. Let $X = (0,1)^2 \ni (\alpha, \beta)$ and assume that $1, \alpha, \beta$ are linearly independent over $\boldsymbol{Q}$. Put

$$a(\alpha, \beta) := \left[\frac{1}{\alpha}\right],\ b(\alpha, \beta) := \left[\frac{\beta}{\alpha}\right]$$

and

$$T(\alpha, \beta) := \left(\frac{\beta}{\alpha} - b(\alpha, \beta),\ \frac{1}{\alpha} - a(\alpha, \beta)\right),$$

then T is a transformation of X into itself. $(X, T, a(\alpha, \beta), b(\alpha, \beta))$ is called *Jacobi-Perron algorithm.* Put

$$(a_n, b_n) := (a(T^{n-1}(\alpha, \beta)),\ b(T^{n-1}(\alpha, \beta)))\quad n = 1,2,\cdots\ ,$$

$$A(a, b) := \begin{pmatrix} a & 0 & 1 \\ 1 & 0 & 0 \\ b & 1 & 0 \end{pmatrix}$$

and

$$\begin{pmatrix} q_n & q_{n-2} & q_{n-1} \\ p_n & p_{n-2} & p_{n-1} \\ r_n & r_{n-2} & r_{n-1} \end{pmatrix} := A(a_1, b_1) \cdots A(a_n, b_n)\ n = 1,2,\cdots,$$

then it is shown that

$$\max(|\alpha - p_n/q_n|, |\beta - r_n/q_n|) \to 0 \quad \text{as } n \to \infty.$$

$(p_n/q_n, r_n/q_n)$ is called the n-th *convergent* of (α, β). If

$$T^m(\alpha, \beta) = T^n(\alpha, \beta) \quad \text{for } m \neq n,$$

then (α, β) is called *periodic.*

Suppose now (α, β) is periodic and let

$$T^m(\alpha, \beta) = T^{m+p}(\alpha, \beta) \quad p \geqq 1.$$

Put

$$M = A(a_{m+1}, b_{m+1}) \cdots A(a_{m+p}, b_{m+p}).$$

M is a 3×3 integral matrix. Perron [1] proved the following result.

The following conditions (1), (2) *are equivalent.*

(1) *M has the eigenvalues $\lambda, \lambda_1, \lambda_2$ such that*

$$\begin{cases} \lambda \in \boldsymbol{R} \quad \lambda > 1 \\ \lambda_1, \lambda_2 : \textit{imaginary} \quad |\lambda_1| = |\lambda_2| < 1 \end{cases}$$

and the column vector ${}^t(1, \alpha, \beta)$ is the eigenvector for λ.

(2) *The order of approximations of (α, β) by the convergents $(p_n/q_n, r_n/q_n)$ is of exponent $1/2$, i.e. for some $K > 0$*

$$\sqrt{q_n}\,|q_n\alpha - p_n| < K,\quad \sqrt{q_n}\,|q_n\beta - r_n| < K \quad \textit{for all } n \in \boldsymbol{N}.$$

It is not difficult to see that if (α, β) is periodic and the condition (1) is satisfied, then for some K