12. *Orders in Quadratic Fields. I*

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Abstract: The primary purpose herein is to provide sufficient conditions for a quadratic order to have its class group generated by ambiguous ideals, and we conjecture that the conditions are in fact necessary. These conditions are given in terms of certain prime-producing quadratic polynomials.

Key words and phrases: Real quadratic order; class Group; quadratic polynomial.

§1. Notation and preliminaries. Let $[\alpha, \beta]$ denote the $\boldsymbol{Z}$-module $\{\alpha x + \beta y : x, y, \in \boldsymbol{Z}\}$ and fix $D_0 \in \boldsymbol{Z}$ as a (positive or negative) square-free integer. Set $\sigma = 2$ if $D_0 \equiv 1 \pmod 4$ and $\sigma = 1$ otherwise. Define $\omega_0 = (\sigma - 1 + \sqrt{D_0})/\sigma$, $\Delta_0 = (\omega_0 - \bar{\omega}_0)^2 = 4\, D_0/\sigma^2$, where $\bar{\omega}_0$ is the algebraic conjugate of ω_0, and let $\omega_\Delta = f\omega_0 + h$ where $f, h \subset \boldsymbol{Z}$. If we set $\mathcal{O}_\Delta = [1, f\omega_0] = [1, \omega_\Delta]$ and $\Delta = (\omega_\Delta - \bar{\omega}_\Delta)^2 = f^2\Delta_0$ then $\mathcal{O}_\Delta$ is an order in $Q(\sqrt{\Delta}) = Q(\sqrt{D_0})$ having *conductor f*, and *fundamental discriminant* Δ_0. Moreover D_0 is the *radicand*; i.e., the square-free kernel of the discriminant Δ. It is well-known (e.g. see [1]) that I is a non-zero ideal in $\mathcal{O}_\Delta$ if and only if $I = [a, b + c\omega_\Delta]$ where $a, b, c, \in \boldsymbol{Z}$ with $c \mid b$, $c \mid a$, and $ac \mid N(b + c\omega_\Delta)$, where N is the norm from $Q(\sqrt{\Delta})$ to Q; i.e., $N(\alpha) = \alpha\bar{\alpha}$. I is called *primitive* if $c = 1$, and $a > 0$. In this case a is the smallest positive integer in I and $a = N(I) = (\mathcal{O}_\Delta : I)$. A primitive ideal I can be written as $I = [a, b + \omega_\Delta]$ with $0 \le b \le a$. An ideal I in $\mathcal{O}_\Delta$ is called *regular* if $\mathcal{O}_0 = \{\alpha \in Q(\sqrt{\Delta}) : \alpha I \subseteq I\}$. All regular ideals are *invertible*. Note that an ideal I is invertible if there is an element $\gamma \in I$ such that $\gcd(f, N(\gamma)) = 1$, (e.g. see [1, Theorem 7, p.122]). Thus if $\gcd(f, N(I)) = 1$ then I is invertible. We denote *equivalence of ideals* by $I \sim J$ (by which we mean that there are non-zero elements α_1 and α_2 of $\mathcal{O}_\Delta$ with $\alpha_1 I = \alpha_2 J$), and we denote the group of equivalence classes by C_Δ (and note that $C_\Delta \cong \mathrm{Pic}\,\mathcal{O}_\Delta$). Let h_Δ be the order of C_Δ; i.e., the *class number* of $\mathcal{O}_\Delta$. We denote the *exponent* of C_Δ by e_Δ; i.e., the smallest positive integer e_Δ such that $I^{e_\Delta} \sim 1$ for all I in C_Δ. Also *principal* ideals generated by a single element α are denoted by (α). We denote finally

$$M_\Delta = \begin{cases} \sqrt{-\Delta/3} & \text{if } \Delta < 0 \\ \sqrt{\Delta/5} & \text{if } \Delta > 0. \end{cases}$$

The following is well-known, (e.g. see [1, Theorem 11, p.141]).

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