95. *Elliptic Factors of Selberg Zeta Functions*

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We show that the elliptic factors of Selberg zeta functions are expressed in terms of multiple gamma functions.

§1. Elliptic factors. Let $X = G/K$ be a rank one symmetric space of non compact type, where G is a connected semisimple Lie group with finite center, and K is a maximal compact subgroup of G. Put $\mathfrak{g} = \mathrm{Lie}(G)$, $\mathfrak{k} = \mathrm{Lie}(K)$ and let $\mathfrak{g}_C$, $\mathfrak{k}_C$ be their complexifications. We assume that $\mathrm{rank}G = \mathrm{rank}K$ and fix a Cartan subgroup T of G which is contained in K. We choose a system of positive roots of $\Phi(\mathfrak{g}_C, \mathfrak{t}_C)$ and a singular imaginary root α_t in $\Phi^+(\mathfrak{g}_C, \mathfrak{t}_C)$. Let $G = KA_RN$ be an Iwasawa decomposition of G. From the assumption, A_R is a one dimensional real torus. We identify $\mathfrak{a}_R$ with $\boldsymbol{R}$ as in [11]. Let ρ_0 be the half of the sum of positive roots in $\Phi(\mathfrak{g}, \mathfrak{a}_R)$. Let M be the centralizer of A_R in K, and A_I be a Cartan subgroup of M. Let $\mathfrak{m}$, $\mathfrak{a}_R$, $\mathfrak{a}_I$ be the Lie algebras of M, A_R, A_I, and $\mathfrak{m}_C$, $\mathfrak{a}_{R,C}$, $\mathfrak{a}_{I,C}$ their complexifications respectively. Let Γ be a discrete subgroup of G such that $vol(G/\Gamma) < \infty$. We define the elliptic factor of the Selberg zeta function for (G, Γ) as a smooth function $Z_{ell}(s)$ on a half interval (a, ∞) of $\boldsymbol{R}$ which satisfies the identity

$$(*)\qquad \left(-\frac{1}{2(s-\rho_0)}\frac{d}{ds}\right)^m \log Z_{ell}(s) = \int_0^{+\infty} I_{ell}(h_t)e^{-s(s-2\rho_0)t}t^{m-1}dt$$

for a positive integer m, where I_{ell} denotes the elliptic term of the Selberg trace formula for (G, Γ), and h_t is the spherical fundamental solution of the heat equation $\left(\Delta + \frac{\partial}{\partial t}\right)u = 0$ on X. By this definition $Z_{ell}(s)$ is determined up to a factor $exp(P(s-\rho_0))$, where $P(s)$ is an even polynomial. We calculate the right hand side of $(*)$ using the Fourier inversion formula of elliptic orbital integrals [11], and determine $Z_{ell}(s)$ as a finite product of multiple gamma functions.

§2. Results. Let $\mathscr{E}_\Gamma$ be the set of elliptic conjugacy classes of Γ, consisting of all conjugacy classes of finite orders. For $\gamma \in \mathscr{E}_\Gamma$, we denote by n_γ its order. We choose an element t_γ of T which is conjugate to γ in G; t_γ is unique up to the action of the Weyl group $W = W(G, T)$. Let G_γ be the centralizer of t_γ in G and $\mathfrak{g}_\gamma$ be its Lie algebra. We write Φ_γ^+, Φ_I^+ the sets of all positive roots in $\Phi(\mathfrak{g}_{\gamma,C}, \mathfrak{t}_C)$, $\Phi_I = \Phi(\mathfrak{m}_C, \mathfrak{a}_{I,C})$ and r_γ, r_I their cardinalities respectively. For each element $w \in W$, put

$$P_{\gamma,w}(\nu) = \prod_{\beta\in\Phi_\gamma^+}(w(-\rho_I + \nu\alpha_t), \beta) \quad (\nu \in \boldsymbol{R}),$$

where ρ_I is the half of the sum of roots in Φ_I^+. For an integral $\lambda \in \sqrt{-1}\mathfrak{t}^*$,