89. Q-rationality of Moment Maps

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The purpose of this note is to announce our recent results (see Theorems A, B and C) associated with the $\boldsymbol{Q}$-rationality of moment maps.

Let X be a compact complex connected manifold carrying a Kähler class κ in $H^2(X, \boldsymbol{Q})$. Then we can choose a very ample line bundle L satisfying $c_1(L) = m\kappa$ for some positive integer m, so that we can regard X as a projective algebraic manifold. Put $n := \dim_{\boldsymbol{C}} X$. Assume further that X admits an effective biregular action of the r-dimensional algebraic torus

$$G = \boldsymbol{G}_{\mathrm{m}}^r = \{(z_1, z_2, \ldots, z_r)\ ;\ z_\alpha \in \boldsymbol{C}^* \text{ for all } \alpha\}.$$

Let $\mathfrak{g} = \sum_{\alpha=1}^r \boldsymbol{C}\mathscr{X}_\alpha$ be the Lie algebra of G, where $\mathscr{X}_\alpha := \sqrt{-1}\, z_\alpha \partial/\partial z_\alpha$. For the maximal compact subgroup $G_{\boldsymbol{R}} \cong (S^1)^r$ of G, consider the associated real Lie subalgebra $\mathfrak{g}_{\boldsymbol{R}} = \sum_{\alpha=1}^r \boldsymbol{R}\mathscr{X}_\alpha$ of $\mathfrak{g}$. Moreover, $\mathfrak{g}$ has a natural $\boldsymbol{Q}$-structure by $\mathfrak{g}_{\boldsymbol{Q}} = \sum_{\alpha=1}^r \boldsymbol{Q}\mathscr{X}_\alpha$. Take a $G_{\boldsymbol{R}}$-invariant Kähler form ω on X in the class κ. Now, to each $\mathscr{Y} \in \mathfrak{g}_{\boldsymbol{R}}$, we can uniquely associate a Hamiltonian function $\mu_\omega^{\mathscr{Y}}$ on X such that

$$\bar{\partial}\mu_\omega^{\mathscr{Y}} = i_{\mathscr{Y}}(2\pi\omega),$$

where $\mu_\omega^{\mathscr{Y}}$ is real-valued and is required to satisfy the normalization condition $\int_X \mu_\omega^{\mathscr{Y}}\, \omega^n = 0$. Let $\mu_\omega : X \to \mathfrak{g}_{\boldsymbol{R}}^*$ be the moment map defined by setting

$$\langle \mu_\omega(x), \mathscr{Y} \rangle = \mu_\omega^{\mathscr{Y}}(x), \quad \mathscr{Y} \in \mathfrak{g}_{\boldsymbol{R}},$$

for each $x \in X$. This moment map is intrinsic in the sense that it is free from any ambiguity of translation caused by the choice of a G-linearization (cf. Mumford and Forgarty [6]) of a power of L. Let $\{\mathscr{X}_1^*, \ldots, \mathscr{X}_r^*\}$ be the $\boldsymbol{R}$-basis for $\mathfrak{g}_{\boldsymbol{R}}^*$ dual to $\{\mathscr{X}_1, \ldots, \mathscr{X}_r\}$ for $\mathfrak{g}_{\boldsymbol{R}}$. The $\boldsymbol{R}$-basis $\{\mathscr{X}_1^*, \ldots, \mathscr{X}_r^*\}$ allows us to identify $\mathfrak{g}_{\boldsymbol{R}}^*$ with $\boldsymbol{R}^r$, so that μ_ω is rewritten as follows:

$$\mu_\omega(x) = (\mu_\omega^{\mathscr{X}_1}(x), \mu_\omega^{\mathscr{X}_2}(x), \ldots, \mu_\omega^{\mathscr{X}_r}(x)) \in \boldsymbol{R}^r, \quad x \in X.$$

Note the following standard fact (due to Atiyah [1], Guillemin and Sternberg [4]) that the image $\mu_\omega(X)$ of the moment map μ_ω is the convex hull of the finite set $\mu_\omega(X^G)$ in $\boldsymbol{R}^r$, where X^G denotes the fixed point set of the G-action on X. Note also that $\dim_{\boldsymbol{R}} \mu_\omega(X) = r$. Let $\mathrm{Crt}(\mu_\omega)$ be the set of all critical values for μ_ω. As in the case [4] of a moment map (which differs from our intrinsic μ_ω by a translation) associated with a G-linearization of a power of L, we have the following:

Theorem A. *The finite subset $\mu_\omega(X^G)$ of $\boldsymbol{R}^r$ sits in $\boldsymbol{Q}^r$. Moreover, $\boldsymbol{R}^r$ naturally admits a finite number of real linear subspaces $H_1, H_2, \ldots, H_p$, all defined over $\boldsymbol{Q}$ and not necessarily passing through the origin, such that*

(1) $\mathrm{Crt}(\mu_\omega) \subset \cup_{j=1}^p H_j$;