# 7. *A Remark on Higher Circular l-Units*

By Yasutaka IHARA
Research Institute for Mathematical Sciences, Kyoto University

(Communicated by Shokichi IYANAGA, M. J. A., Jan. 13, 1992)

**1.** Let $l$ be a prime number, and $E_l=E(\{0,1,\infty\})$ be the group of higher circular $l$-units defined and studied in [1] [2] (esp. [1] §2·6). As is shown in [1], elements of $E_l$ are *l-units* in the maximal pro-$l$ extension $M_l$ of $\boldsymbol{Q}(\mu_{l^\infty})$ unramified outside $l$ ($\mu_{l^\infty}$: the group of $l$-powerth roots of 1), and $\boldsymbol{Q}(E_l)$ corresponds to the kernel of the canonical representation of the Galois group $\mathrm{Gal}(\bar{\boldsymbol{Q}}/\boldsymbol{Q})$ in the outer automorphism group of the pro-$l$ fundamental group of $\boldsymbol{P}^1-\{0,1,\infty\}$. The main purpose of this note is to prove the following

**Theorem.** *For any $\varepsilon\in E_l$ and $\sigma\in\mathrm{Gal}(\bar{\boldsymbol{Q}}/\boldsymbol{Q})$, $\varepsilon^{\sigma-1}$ is a unit.*

In other words, if $\varepsilon\in E_l$ and $k$ is any finite Galois extension over $\boldsymbol{Q}$ containing $\varepsilon$, then the fractional ideal $(\varepsilon)=\varepsilon\mathcal{O}_k$ is $\mathrm{Gal}(k/\boldsymbol{Q})$-invariant ($\mathcal{O}_k$: the ring of integers of $k$).

The above theorem holds trivially when $l$ is a *regular* prime. In fact, in this case, $l$ has a unique extension in $M_l$ and hence every $l$-unit in $M_l$ has the claimed property. (To see that $l$ has a unique extension in $M_l$, first observe that it is so in the maximal $l$-elementary abelian extension of $\boldsymbol{Q}(\mu_l)$ unramified outside $l$; then apply the Burnside principle "a closed subgroup $D$ of a pro-$l$ group $G$ coincides with $G$ if its image $\bar{D}$ on the Frattini quotient $\bar{G}$ of $G$ coincides with $\bar{G}$" to the decomposition group $D\subset\mathrm{Gal}(M_l/\boldsymbol{Q}(\mu_l))$ of an extension of $l$.) But when $l$ is *irregular*, $l$ does decompose in $M_l$; hence not all the $l$-units of $M_l$ can enjoy the property stated in the theorem.

In [1] (§0·2), we raised two questions (a) (b), which, in the present language, read as

(a) $\boldsymbol{Q}(E_l)=M_l$?

(b) *Is $E_l$ the full group of l-units in $\boldsymbol{Q}(E_l)$*?

The above theorem implies that when $l$ is irregular, $E_l$ cannot be the group of all $l$-units in $M_l$, and hence at most one of (a) (b) can have an affirmative answer. In any case, it is an interesting open question to characterize the field $\boldsymbol{Q}(E_l)$ and the group $E_l$.

**2. Proof of the theorem.** The proof is quite elementary. Let $v$ denote any extension to $\bar{\boldsymbol{Q}}$ of the normalized additive $l$-adic valuation $\mathrm{ord}_l$ of $\boldsymbol{Q}$ (so, $v(l)=1$).

**Lemma 1.** *If $a=b^l\in\bar{\boldsymbol{Q}}^\times$ and $v(a-1)<l(l-1)^{-1}$, then $v(b-1)=l^{-1}\times v(a-1)$.*

*Proof.* Decompose $a-1$ into the product of $b-\zeta^i$ over all $i(\mathrm{mod}\ l)$, $\zeta$