26. *Some Observations Concerning the Distribution of the Zeros of the Zeta Functions. III*

By Akio FUJII
Department of Mathematics, Rikkyo University

(Communicated by Shokichi IYANAGA, M. J. A., May 12, 1992)

§ 1. **Introduction.** The purpose of the present article is to give a refinement on our previous results (e.g. Theorems 6 and 7 in pp. 141–142 of [4] and a special case of Theorem in p. 100 of [1]) on the exponential sums over the zeros of some zeta functions. Some of the consequences resulting from this will also be mentioned.

Let $L(s,\psi)$ be a Dirichlet L-function with a primitive Dirichlet character $\psi \bmod k\geq 1$. When $k=1$, we suppose that $L(s,\psi)=\zeta(s)=$ the Riemann zeta function. Let $\gamma(\psi)$ run over the imaginary parts of the zeros of $L(s,\psi)$. We shall prove the following theorem under the Generalized Riemann Hypothesis (G.R.H.) for $L(s,\psi)$.

Theorem. (Under G.R.H.) *Suppose that $0<\alpha\ll T$, $1\ll Y\leq T$ and b be any positive number. Then we have*

$$\sum_{Y<\gamma(\psi)\leq T} e^{ib\gamma(\psi)\log(b\gamma(\psi)/2\pi e\alpha)}=-e^{(\pi/4)i}\frac{\sqrt{\alpha}}{b}\sum_{(Yb/2\pi\alpha)^b\leq n\leq (Tb/2\pi\alpha)^b}\frac{\Lambda(n)\psi(n)}{n^{1/2-1/2b}}e^{-2\pi i\alpha n^{1/b}}$$
$$+O\Big(\mathrm{Min}\Big(\frac{1}{|\log Y/\alpha|},\sqrt{\alpha}+1\Big)\log kT\Big)+O\Big(\frac{\log kT}{\log\log kT}\Big)$$
$$+O(B(b,k,\alpha,T))+O\Big(C(b,\alpha,Y,T)\log\frac{T}{\alpha}\Big),$$

where $\Lambda(n)$ is the von-Mangoldt function,

$$C(b,\alpha,Y,T)=\begin{cases}\left(\dfrac{\alpha}{Y}\right)^{b/2}\sqrt{Y} & \text{if } b>1\\ \sqrt{\alpha}\log\dfrac{T}{Y} & \text{if } b=1\\ \eta_\alpha(Y)\sqrt{T}\left(\dfrac{\alpha}{T}\right)^{b/2}+(1-\eta_\alpha(Y))\sqrt{\alpha} & \text{if } 0<b<1,\end{cases}$$

$\eta_\alpha(Y)=1$ if $\alpha<CY$, and $=0$ if $\alpha\geq CY$ with some positive constant C and $B(b,k,\alpha,T)$

$$=\mathrm{Min}\left(\left(\frac{T}{\alpha}\right)^{b/2}\log\frac{T}{\alpha}\,\frac{\log kT}{(\log\log kT)^2},\ \left(\frac{T}{\alpha}\right)^{b/(\log\log kT)}\frac{\log kT}{\log((\log kT)/(T/\alpha)^b+2)}\right)$$
$$+\left(\frac{T}{\alpha}\right)^{b/2}\left(\log\frac{T}{\alpha}\log\log\frac{T}{\alpha}+\sqrt{\frac{\log kT}{\log\log kT}}\,\frac{1}{\log((T/\alpha)^b/(\log kT\cdot\log\log kT)+2)}\right.$$
$$\left.+\frac{\log(T/\alpha)}{T}\,\frac{\log kT}{(\log\log kT)^2}\right).$$

If we ignore the dependence on k and α, we get the following simpler