23. *Certain Integral Operators*[*)]

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1. Introduction. Let $\mathcal{A}(p)$ be the class of functions of the form

$$f(z)=z^p+\sum_{k=p+1}^{\infty} a_k z^k \qquad (p\in\mathcal{N}=\{1,2,3,\cdots\}) \tag{1.1}$$

which are analytic in the unit disk $\mathcal{U}=\{z:|z|<1\}$. For $f(z)\in\mathcal{A}(p)$, we define

$$I_0 f(z)=\left(\frac{f(z)}{z^p}\right)^{\alpha} \qquad (\alpha>0) \tag{1.2}$$

and

$$I_n f(z)=\frac{1}{z}\int_0^z I_{n-1}f(t)dt \qquad (n\in\mathcal{N}). \tag{1.3}$$

For $f(z)$ belonging to the class $\mathcal{A}(1)$, Thomas [4] has shown

Theorem A. *If $f(z)\in\mathcal{A}(1)$ satisfies*

$$\mathrm{Re}\left\{f'(z)\left(\frac{f(z)}{z}\right)^{\alpha-1}\right\}>0 \qquad (z\in\mathcal{U}) \tag{1.4}$$

for some α $(\alpha>0)$, then

$$\mathrm{Re}\,(I_n f(z))\geqq\gamma_n(r)>\gamma_n(1), \tag{1.5}$$

where $n\in\mathcal{N}_0=\{0,1,2,\cdots\}$ and

$$0<\gamma_n(r)=-1+2\alpha\sum_{k=1}^{\infty}\frac{(-1)^{k+1}r^{k-1}}{k^n(k-1+\alpha)}<1. \tag{1.6}$$

Equality occurs for the function $f(z)$ defined by

$$f(z)=\left(\alpha\int_0^z t^{\alpha-1}\left(\frac{1-t}{1+t}\right)dt\right)^{1/\alpha}. \tag{1.7}$$

For $n=0$, (1.5) becomes

$$\mathrm{Re}\left\{\left(\frac{f(z)}{z}\right)^{\alpha}\right\}\geqq\frac{\alpha}{r^\alpha}\int_0^r t^{\alpha-1}\left(\frac{1-t}{1+t}\right)dt \tag{1.8}$$

$$=-1+2\alpha\sum_{k=1}^{\infty}\frac{(-1)^{k+1}r^{k-1}}{k-1+\alpha},$$

which reduces to

$$-1+\frac{2}{r}\log(1+r)$$

when $\alpha=0$.

Also, Hallenbeck [1] has proved

Theorem B. *If $f(z)\in\mathcal{A}(1)$ satisfies*

$$\mathrm{Re}\{f'(z)\}>0 \qquad (z\in\mathcal{U}), \tag{1.9}$$

then

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