85. On Fundamental Units of Real Quadratic Fields with Norm -1

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1. There are many known results on explicit forms of the fundamental units of real quadratic fields of certain types (cf. [1], [2], [3], [5], [7], [8]). In this paper, we shall give a new explicit form of the fundamental units of real quadratic fields with norm -1. Let m be a positive integer which is not a perfect square and K be the real quadratic field $\boldsymbol{Q}(\sqrt{m})$. ε_0 denotes the fundamental unit of K, N the norm map from K to $\boldsymbol{Q}$. We put

$$R_-=\{K: \textit{real quadratic fields with } N\varepsilon_0=-1\},$$
$$E_-=\{\varepsilon: \textit{units of } K\in R_- \textit{ with } N\varepsilon=-1\}.$$

Then it is easy to see $R_-=\{\boldsymbol{Q}(\sqrt{a^2+4}): a\in \boldsymbol{N}\}$, where $\boldsymbol{N}$ is the set of all the natural numbers. Fix now a unit $\varepsilon=(t+u\sqrt{m})/2\in E_-$ $(t, u>0)$ for a while, and we denote $\varepsilon^n=(t_n+u_n\sqrt{m})/2$. $\bar{\varepsilon}$ denotes $(t-u\sqrt{m})/2$. Since $t_n=\varepsilon^n+\bar{\varepsilon}^n$, we have

$$t_{n+1}=\varepsilon^{n+1}+\bar{\varepsilon}^{n+1}=(\varepsilon^n+\bar{\varepsilon}^n)(\varepsilon+\bar{\varepsilon})+\varepsilon^{n-1}+\bar{\varepsilon}^{n-1}=tt_n+t_{n-1} \quad (n\geq 2).$$

Combining this recurrence and the fact $t_1=t$ and $t_2=t^2+2$, we get inductively $t\mid t_{2n+1}$ and $t_{2n+1}\geq t_3\geq 4t$ $(n\geq 1)$. Hence we have obtained the following elementary lemma.

Lemma 1. *With the above notation, we have*

(i) $t_{n+1}=tt_n+t_{n-1}$ $(n\geq 2)$ *and* $t_1=t$, $t_2=t^2+2$,

(ii) $t\mid t_{2n+1}$ and $t_{2n+1}\geq 4t$ $(n\geq 1)$.

From this lemma follows:

Lemma 2. *If t_{2n+1} is a prime, then $t=1$ and $2n+1$ is prime.*

Proof. If $t\geq 2$, t_{2n+1} can not be a prime by Lemma 1 (ii). Suppose now $2n+1$ decomposes into $2n+1=(2k+1)(2l+1)$, where $2k+1, 2l+1>1$. Then, from (ii) of Lemma 1, $\varepsilon^{2n+1}=(\varepsilon^{2k+1})^{2l+1}$ implies $t_{2k+1}\mid t_{2n+1}$, $t_{2k+1}\geq 4$ and $t_{2n+1}/t_{2k+1}\geq 4$. Therefore t_{2n+1} can not be a prime.

Examine now the case $t=1$, From $N\varepsilon=-1$ and $t=1$ follows $u^2m=5$, so $u=1$, $m=5$. Thus t_n is nothing but *the nth Lucas number* $v_n=\{(1+\sqrt{5})/2\}^n+\{(1-\sqrt{5})/2\}^n$ (cf. [4]). Let $P_1=\{p$: primes such that $p=v_{2n+1}$, $n\geqq 1\}$. If the set P_1 is infinite or not is a famous open problem, but we shall consider the problem how the set P_1 is distributed in the set of all the primes.

For any $N>0$, we put

$$\rho_1(N)=\text{the number of primes } p \text{ such that } p\in P_1 \text{ and } p\leq N.$$

As usual we put

$$\pi(N)=\text{the number of primes } p \text{ such that } p\leq N.$$