80. On the Starlikeness of the Bernardi Integral Operator

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Abstract: Denote by A the class of functions f analytic in the unit disc D and normalised so that $f(0)=f'(0)-1=0$. For $f\in A$ and $-1<c\leq 0$, let F_c be defined by $F_c(z)=\frac{(1+c)}{z^c}\int_0^z t^{c-1}f(t)dt$ for $z\in D$. We find estimates on β so that $\operatorname{Re} f'(z)>\beta$ will ensure the starlikeness of F_c.

Introduction. Denote by A the class of functions f which are analytic in the unit disc $D=\{z: |z|<1\}$ and normalised so that $f(0)=f'(0)-1=0$. Let R be the subclass of A satisfying $\operatorname{Re} f'(z)>0$ for $z\in D$ and S^* be the subset of starlike functions, i.e.

$$S^*=\left\{f\in A: \operatorname{Re}\frac{zf'(z)}{f(z)}>0 \text{ for } z\in D\right\}.$$

If S denotes the subset of A consisting of univalent functions, then it is well known that $R\subset S$ and $S^*\subset S$. Krzyz [3] gave an example to show that R is not a subset of S^*. On the other hand, Singh and Singh [6] showed that $f\in R$ would imply $F_0\in S^*$, where

$$F_0(z)=\int_0^z \frac{f(t)}{t}\,dt.$$

In a later paper, Singh and Singh [7] showed that $\operatorname{Re} f'(z)>-\frac{1}{4}$ is sufficient to ensure $F_0\in S^*$ and more recently [5] it was shown that $\operatorname{Re} f'(z)>-0.262$ implies the same.

Suppose that $f\in A$ and $c>-1$. For $z\in D$, the Bernardi operator [1] is defined by

$$F_c(z)=\frac{1+c}{z^c}\int_0^z t^{c-1}f(t)dt. \tag{1}$$

It was shown in [5] that $\operatorname{Re} f'(z)>\beta$ implies $F_c\in S^*$ provided

$$(1+c)\beta>\frac{\log(4/e)}{6}\left(c^2\tan^2\frac{\alpha^*\pi}{2}-3\right) \tag{2}$$

where $1=\alpha^*+(2/\pi)\tan^{-1}\alpha^*$. We note that when $c=0$, $\beta=-0.193$ which is not as good an estimate as the constant -0.262. If $c=1$, then $\beta=-0.017$, which was also obtained in [4].

In this paper we shall improve the constant β in (2) for $-1<c\leq 0$.

Results. Theorem. Suppose $f\in A$ and F_c be given by (1) and that

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