76. On the Asymptotic Remainder Estimate for the Eigenvalues of Operators Associated with Strongly Elliptic Sesquilinear Forms

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§ 1. **Introduction and main result.** This present note is devoted to the supplementary result to be added to the previous paper [5].

Let Ω be a bounded domain in the n-dimensional Euclidean space $\boldsymbol{R}^n$. For a nonnegative integer m and $p>1$ we denote by $W_p^m(\Omega)$ with the norm $\| \ \|_{m,p}$ the space of functions whose distributional derivatives of order up to m belong to $L_p(\Omega)$, and by $W_{p,0}^m(\Omega)$ the closure of $C_0^\infty(\Omega)$ in $W_p^m(\Omega)$. In particular we set $H^m(\Omega)=W_2^m(\Omega)$, $\| \ \|_m=\| \ \|_{m,2}$ and $H_0^m(\Omega)=W_{2,0}^m(\Omega)$. Let B be an integro-differential symmetric sesquilinear form of order m with bounded coefficients:

$$B[u,v]=\int_\Omega \sum_{|\alpha|,|\beta|\leqq m} a_{\alpha\beta}(x)D^\alpha u(x)\overline{D^\beta v(x)}dx,$$

$$\alpha=(\alpha_1,\cdots,\alpha_n),\quad D^\alpha=(-\sqrt{-1})^{|\alpha|}(\partial/\partial x_1)^{\alpha_1}\cdots(\partial/\partial x_n)^{\alpha_n},$$

which is coercive on $H_0^m(\Omega)$:

$$B[u,u]\geqq\delta\|u\|_m^2-C_0\|u\|_0^2,\quad \delta>0,\quad C_0\geqq 0\quad \text{for any } u\in H_0^m(\Omega).$$

Let A be the operator associated with the variational triple $\{B, H_0^m(\Omega), L_2(\Omega)\}$. That is, $u\in H_0^m(\Omega)$ belongs to $D(A)$, the domain of A if and only if there exists $f\in L_2(\Omega)$ such that $B[u,v]=(f,v)_{L_2(\Omega)}$ for any $v\in H_0^m(\Omega)$ and we define $Au=f$. As is known, A is a self-adjoint operator and the spectrum of A consists of eigenvalues accumulating only at $+\infty$. For a real number t let $N(t;A)$ or simply $N(t)$ denote the number of eigenvalues of A not exceeding t. We put

$$a(x,\xi)=\sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}(x)\xi^{\alpha+\beta},$$

$$\mu_A(x)=(2\pi)^{-n}\int_{a(x,\xi)<1} d\xi,\quad \mu_A(\Omega)=\int_\Omega \mu_A(x)dx.$$

For $\tau=k+\sigma>0$ with an integer k and $0<\sigma\leqq 1$ let $\mathscr{B}^\tau(\Omega)$ denote the space of functions u in Ω such that $D^\alpha u$ are bounded and continuous for $|\alpha|\leqq k$ and $|D^\alpha u(x)-D^\alpha u(y)|/|x-y|^\sigma$ $(x,y\in\Omega, x\neq y)$ are bounded for $|\alpha|=k$.

In [5] we investigated the remainder estimate in the asymptotic formula for the eigenvalues of A with $a_{\alpha\beta}\in\mathscr{B}^\tau(\Omega)$ $(|\alpha|=|\beta|=m)$ for $\tau>0$. But we could not give any assertion for $0<\tau<m$ when $2m\leqq n$. In this note we settle this case.

Theorem. *Let $\tau>0$. Suppose that $a_{\alpha\beta}\in\mathscr{B}^\tau(\Omega)$ $(|\alpha|=|\beta|=m)$ and that the boundary $\partial\Omega$ is in C^{2m}-class. Then we have*

$$N(t)=\mu_A(\Omega)t^{n/2m}+O(t^{(n-\theta)/2m})\quad \textit{as } t\to\infty,$$