64. An Additive Problem of Prime Numbers. II

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1. Introduction. We continue our study [1] on the mean value of

$$r_2(n)=\sum_{m+k=n}\Lambda(m)\Lambda(k),$$

where $\Lambda(x)=\log p$ if $x=p^m$ with a prime number p and an integer $m\geq 1$, and $=0$ otherwise. We put

$$S_2(n)=\prod_{p\mid n}\left(1+\frac{1}{p-1}\right)\prod_{p\nmid n}\left(1-\frac{1}{(p-1)^2}\right).$$

Then in [1] we have shown under the Riemann Hypothesis (R.H.) that

$$\sum_{n\leq X}(r_2(n)-nS_2(n))=O(X^{3/2}).$$

Here we shall clarify the oscillating nature of the right hand side of this formula. We shall carry it out by expressing several places of the previous arguments in [1] more explicitly. As a result we shall prove the following theorem.

Theorem (Under R.H.). *For $X>X_0$, we have*

$$\sum_{n\leq X}(r_2(n)-nS_2(n))=-4X^{3/2}\Re\left[\sum_{\gamma>0}\frac{X^{i\gamma}}{((1/2)+i\gamma)((3/2)+i\gamma)}\right]+O((X\log X)^{1+(1/3)}),$$

where γ runs over the imaginary parts of the zeros of the Riemann zeta function $\zeta(s)$.

As is seen below, another oscillating nature connected with the distribution of the zeros of $\zeta(s)$ is hidden in the remainder term

$$O((X\log X)^{1+(1/3)}),$$

although the bounded quantity

$$G(X)\equiv\Re\left[\sum_{\gamma>0}\frac{X^{i\gamma}}{((1/2)+i\gamma)((3/2)+i\gamma)}\right]$$

represents the main oscillation.

We assume R.H. throughout this article.

Some information concerning the quantity $G(X)$ will be noticed in the forthcoming article [2].

2. Proof of Theorem. We put

$$R(y)=\sum_{n\leq y}\Lambda(n)-y \qquad \text{for } y\geq 0.$$

We put $N=[X]$. Then we have

$$\begin{aligned}\sum_{n\leq X}r_2(n)&=\sum_{m\leq X}\Lambda(m)(X-m)+\sum_{2\leq m\leq X-2}\Lambda(m)(R(X-m)-R(N-m))\\&\quad+\sum_{2\leq m\leq N-2}\Lambda(m)R(N-m)+O(\log X)\\&=S_1+S_2+S_3+O(\log X),\text{ say.}\end{aligned}$$

We get first