25. On the Fundamental Groups of Moduli Spaces of Irreducible $SU(2)$-Connections over Closed 4-Manifolds

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§1. **Introduction and statement of result.** Let M be a connected oriented closed smooth four-manifold and $P \to M$ be a principal $SU(2)$ bundle over M with $c_2(P)=k$. Let $E=P\times_{SU(2)} C^2$ be the C^2-vector bundle associated with P by the standard representation, and $AdP=P\times_{Ad} su(2)$ be the $su(2)$ bundle associated with P by the adjoint representation. We fix integers $p\geq 2$ and $l\geq 1$. We set

$$\mathcal{A}_k := \{A+a \mid A \text{ is a smooth connection on } P.\ a\in L_l^p\Omega^1(AdP)\}$$

which is the L_l^p-completion space of the principal connections on P. Here L_l^p means the Sobolev space of sections whose derivatives of order $\leq l$ are bounded in L^p-norms, and we denote the space of AdP valued smooth m-forms on M by $\Omega^m(AdP)$. We set

$$\mathcal{G}_k := C^0(M, P\times_{Ad} SU(2)) \cap L_{l+1}^p\Omega^0(EndE)$$

which is the L_{l+1}^p-completion space of gauge group of P. We denote by $\mathcal{A}_k^*$ the subspace of irreducible connections of $\mathcal{A}_k$. We put $\mathcal{B}_k=\mathcal{A}_k/\mathcal{G}_k$ and $\mathcal{B}_k^*=\mathcal{A}_k^*/\mathcal{G}_k$. We call $\mathcal{B}_k^*$ the moduli space of irreducible $SU(2)$-connections on P. We note that $\mathcal{G}_k$ acts on $\mathcal{A}_k^*$ not freely.

In this note we study the fundamental group of $\mathcal{B}_k^*$. We shall show the following theorem.

Theorem. *Let M be a closed 4-manifold as above. Suppose that M is simply connected.*

(1) *When the intersection form of M is of odd type, then*

$$\pi_1(\mathcal{B}_k^*)=1$$

(2) *When the intersection form of M is of even type, then*

$$\pi_1(\mathcal{B}_k^*)=\begin{cases} 1 & \text{if } c_2(P)=k \text{ is odd.} \\ Z_2 & \text{if } c_2(P)=k \text{ is even.} \end{cases}$$

It is well known that S. K. Donaldson investigated the topology of 4-manifolds by using gauge theory (e.g. [2], [3]). In his works he studied the moduli space $\mathcal{M}_k$ of anti-self dual connections on P with $c_2(P)=k$. Many properties of the topology of $\mathcal{M}_k$ are got by the analysis of anti-self dual equation. But some properties are deduced from that of $\mathcal{B}_k^*$. In fact in [2] we had to show the orientability of $\mathcal{M}_1^*$. We can show it by using the fact that $\mathcal{B}_1^*$ is simply connected ([2], [4]). Further in order to get more refinement invariants of 4-manifolds we shall have to argue with moduli spaces with higher instanton number k. Therefore it is fundamen-