18. A Note on a Paper of Iwasawa

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(Communicated by Shokichi IYANAGA, M. J. A., Feb. 13, 1990)

1. Let F be a finite extension of a finite algebraic number field k and denote by C_k and C_F the ideal class group of k and of F respectively. A subgroup A of C_k is said to capitulate in F if A is contained in the kernel of natural homomorphism $C_k \to C_F$. The principal ideal theorem states that C_k always capitulates in Hilbert's class field K over k. However, for some k, C_k already capitulates in a proper subfield M of K: $k \subseteq M \subsetneqq K$. Such a field M exists if and only if there is a prime number p such that $C_{k,p}$ (=the p-class group of k) capitulates in a proper subfield F of Hilbert's p-class field K_p over k: $k \subseteq F \subsetneqq K_p$ (cf. [1]). In his paper ([1]), Iwasawa gave simple examples of k such that the 2-class group $C_{k,2}$ already capitulates in a proper subfield F of Hilbert's 2-class field K_2 over k.

Iwasawa's example. Let p, p_1, p_2 be 3 distinct prime numbers such that

i) $p \equiv p_1 \equiv p_2 \equiv 1$ mod. 4 and Legendre symbols

$$\left(\frac{p}{p_1}\right)=\left(\frac{p}{p_2}\right)=-1$$

ii) the norm of the fundamental unit of the real quadratic field $k' = \boldsymbol{Q}(\sqrt{p_1 p_2})$ is 1.

Let

$$k=\boldsymbol{Q}(\sqrt{pp_1p_2}), \qquad K_2=\boldsymbol{Q}(\sqrt{p}, \sqrt{p_1}, \sqrt{p_2}),$$
$$F=\boldsymbol{Q}(\sqrt{p}, \sqrt{p_1p_2}).$$

Then K_2 is Hilbert's 2-class field over k and $C_{k,2}$ capitulates in the proper subfield F of K_2: $k \subseteq F \subsetneqq K_2$.

2. Let k and K_2 be as stated above. Then

$$F=\boldsymbol{Q}(\sqrt{p}, \sqrt{p_1p_2}), \quad F_1=\boldsymbol{Q}(\sqrt{p_1}, \sqrt{pp_2}), \quad F_2=\boldsymbol{Q}(\sqrt{p_2}, \sqrt{pp_1})$$

are all proper subfields of K_2 over k. In the following, we shall consider a question whether $C_{k,2}$ capitulates also in F_1 or in F_2.

Proposition 1. *Let $K_2^{(2)}$ denote Hilbert's 2-class field over K_2.*

i) *If $K_2 = K_2^{(2)}$, $C_{k,2}$ capitulates also both in F_1 and in F_2.*

ii) *If $K_2 \neq K_2^{(2)}$, $C_{k,2}$ capitulates neither in F_1 nor in F_2.*

Proof. This is a consequence of Theorem 2 in [2].

Corollary. *$C_{k,2}$ capitulates in $F_1 \Longleftrightarrow C_{k,2}$ capitulates in F_2.*

Proposition 2. *Let $h_2(F)$ be the 2-class number of F. Then*

i) $K_2 = K_2^{(2)} \Longleftrightarrow h_2(F) = 2$.

ii) $K_2 \neq K_2^{(2)} \Longleftrightarrow 4 \mid h_2(F)$.