83. *The Set of Primes Bounded by the Minkowski Constant of a Number Field*

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Let k be an algebraic number field with degree $m=r_1+2r_2\geqq 2$ and discriminant d_k, where (r_1, r_2) denotes the signature of k. Write $M_k=(4/\pi)^{r_2}(m!/m^m)\sqrt{|d_k|}$ (the Minkowski constant of k) and $M(k)=\{p;$ rational prime and $p\leqq M_k\}$. For every prime number p, let $p\,O_k=P_1^{e_1}\cdots P_g^{e_g}$ be the decomposition into prime ideals of O_k (where O_k denotes the ring of integers in k, $P_i\neq P_j$ $(i\neq j)$ are distinct prime ideals of O_k). In general, the prime number p is not necessarily irreducible element in O_k. Let $\mathrm{Irr}(O_k)$ be the set of all irreducible elements in O_k. Now we define nine subsets $A_0(k), A_1(k), \cdots, A_8(k)$ of $M(k)$ as follows.

$A_0(k)=\{p\in M(k);\ g=e_1=1$ (i.e. p remains prime in O_k, so) $p\in \mathrm{Irr}(O_k)\}$

$A_1(k)=\{p\in M(k);\ g=1,\ e_1=m$ (i.e. p is fully ramified), $p\in \mathrm{Irr}(O_k)\}$

$A_2(k)=\{p\in M(k);\ e_1+\cdots+e_g\lneqq m,\ 1\lneqq e_j$ for some j, $p\in \mathrm{Irr}(O_k)\}$

$A_3(k)=\{p\in M(k);\ g=m,\ e_1=\cdots=e_g=1$ (i.e. p splits completely), $p\in \mathrm{Irr}(O_k)\}$

$A_4(k)=\{p\in M(k);\ g\lneqq m,\ e_1=\cdots=e_g=1$ (i.e. p is unramified), $p\in \mathrm{Irr}(O_k)\}$

$A_5(k)=\{p\in M(k);\ g=1,\ e_1=m$ (i.e. p is fully ramified), $p\notin \mathrm{Irr}(O_k)\}$

$A_6(k)=\{p\in M(k);\ e_1+\cdots+e_g\lneqq m,\ 1\lneqq e_j$ for some j, $p\notin \mathrm{Irr}(O_k)\}$

$A_7(k)=\{p\in M(k);\ g=m,\ e_1=\cdots=e_g=1$ (i.e. p splits completely), $p\notin \mathrm{Irr}(O_k)\}$

$A_8(k)=\{p\in M(k);\ g\lneqq m,\ e_1=\cdots=g_g=1$ (i.e. p is unramified), $p\notin \mathrm{Irr}(O_k)\}$.

Then we have $M(k)=A_0(k)\cup A_1(k)\cup\cdots\cup A_8(k)$ (disjoint union). In case $m=2$, the subsets $A_2(k), A_4(k), A_6(k), A_8(k)$ are of course empty.

The following three theorems are variations on the theme of T. Ono [2].

Theorem 1. *If $M(k)=A_0(k)$, then the class number h_k of k is one.*

Proof. By the Minkowski lemma, the ideal class group H_k of k is generated by the classes of prime ideals over $p\in M(k)$. Hence we have $h_k=1$. Q.E.D.

Lemma 1. *Let $aO_k=Q_1\cdots Q_n$ be the decomposition into prime ideals ($Q_1, \cdots, Q_n$ are not necessarily distinct, $a\in O_k$). Suppose that Q_i belongs to an ideal class $x_i\in H_k$ $(1\leqq i\leqq n)$ and x_0 denotes the principal class of H_k. Then a is an irreducible element in O_k if and only if $x_{i_1}\cdots x_{i_m}\neq x_0$ for every proper subset $\{i_1, \cdots, i_m\}$ of $\{1, \cdots, n\}$.*

Proof. See Lemma 1.2 in Czogala [1]. Q.E.D.

Theorem 2. *If $\sharp(A_1(k)\cup A_3(k))\geqq 1$, then $h_k\geqq m=(k:Q)$.*

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