79. On Products of Consecutive Integers

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(Communicated by Shokichi IYANAGA, M. J. A., Dec. 12, 1990)

1. Diophantine equations involving products of integers have been investigated by many mathematicians. Among these are Erdös [1], L. J. Mordell [4], but these are the equations in two variables. In this paper, we shall show the following diophantine equation in three variables

$$x(x+1)y(y+1)=z(z+1) \tag{1}$$

has infinitely many integer solutions and also show there exists an algorithm for obtaining all the integer solutions of (1).

2. In our previous paper [3], we have obtained the following result. We denote the set of all the integer solutions of a diophantine equation $z^2=(x^2-1)(y^2-1)+a$ $(a \in Z)$ by S_a. Then it is easy to verify that the mappings

$$\sigma: \begin{pmatrix} x \\ y \\ z \end{pmatrix} \longrightarrow \begin{pmatrix} x \\ xy+z \\ (x^2-1)y+xz \end{pmatrix}, \qquad \tau: \begin{pmatrix} x \\ y \\ z \end{pmatrix} \longrightarrow \begin{pmatrix} y \\ x \\ z \end{pmatrix},$$

$$\rho_1: \begin{pmatrix} x \\ y \\ z \end{pmatrix} \longrightarrow \begin{pmatrix} -x \\ y \\ z \end{pmatrix}, \quad \rho_2: \begin{pmatrix} x \\ y \\ z \end{pmatrix} \longrightarrow \begin{pmatrix} x \\ -y \\ z \end{pmatrix}, \quad \rho_3: \begin{pmatrix} x \\ y \\ z \end{pmatrix} \longrightarrow \begin{pmatrix} x \\ y \\ -z \end{pmatrix}$$

are the permutations of S_a. G denotes the group $\langle \sigma, \tau, \rho_i \rangle$ $(1 \leqq i \leqq 3)$. We denotes the number of the representatives $\sharp[S_a/G]$ by t_a. Then we have obtained the following proposition.

Proposition (cf. [3]). *The number t_a is finite except in case $a=0$, and t_a equals the number of the integer points contained in the set $S_a \cap R_a$, where*

$$R_a = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : 0 \leqq x \leqq y \leqq \sqrt{(a+1-x^2)/(2x+2)},\ 0 \leqq z \right\} \qquad \textit{in case } a>0,$$

and

$$R_a = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : 1 < x \leqq y,\ 0 \leqq z,\ \sqrt{(x^2-a-1)/(x^2-1)} \leqq y \leqq \sqrt{(x^2-a-1)/(2x-2)} \right\}$$

in case $a<0$.

For the case $a=4$, we have $t_a=2$, that is, $S_a = G\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} \cup G\begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}$. Con-