38. *Solution of a Problem of Yokoi*

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(Communicated by Shokichi IYANAGA, M. J. A., June 12, 1990)

In [12]–[16] Yokoi studied what he called p-invariants for a real quadratic field $Q(\sqrt{p})$ where $p\equiv 1 \pmod 4$ is prime. In [9] we generalized this concept to an arbitrary real quadratic field $Q(\sqrt{d})$ where d is positive and square-free. We provided numerous applications including bounds for fundamental units and an investigation of the class number one problem related to non-zero n_d, (defined below). It is the purpose of this paper to give a complete list and a proof that the list is valid (with one possible value remaining) of all $Q(\sqrt{d})$ having class number $h(d)=1$ when $n_d\neq 0$. Moreover we show that if the exceptional value of d exists then it is a counterexample to the Generalized Riemann Hypothesis. This completes the task of Yokoi begun in [15]–[16].

In what follows the fundamental unit $\varepsilon_d(>1)$ of $Q(\sqrt{d})$ is denoted $(t_d+u_d\sqrt{d})/\sigma$ where $\sigma=\begin{Bmatrix}2 \text{ if } d\equiv 1 \pmod 4\\ 1 \text{ if } d\equiv 2, 3 \pmod 4\end{Bmatrix}$. Now set:

$$B=((2t_d)/\sigma-N(\varepsilon_d)-1)u_d^2$$

where N is norm from $Q(\sqrt{d})$. This boundary B was studied in [4], [5] and [14].

The following generalizes Yokoi's notion of a p-invariant n_p where $p\equiv 1 \pmod 4$ is prime (see [12]–[16]).

Let n_d be the *nearest integer* to B; i.e.,

$$n_d=\begin{Bmatrix}\lfloor B\rfloor & \text{if } B-\lfloor B\rfloor<\frac{1}{2}\\ \lfloor B\rfloor+1 & \text{if } B-\lfloor B\rfloor>\frac{1}{2}\end{Bmatrix}$$

(where $\lfloor x\rfloor$ is the greatest integer less than or equal to x).

In [9] we proved the following:

Theorem 1. *Let $d>0$ be square-free and let $u_d>2$. Then the following are equivalent:*

(1) $n_d=0$

(2) $t_d>4d/\sigma$

(3) $u_d^2>16d/\sigma^2$.

The above generalizes the main result of Yokoi in [12].

We also proved in [9] the following consequences of Theorem 1.

Corollary 1. *If $n_d\neq 0$ then $\varepsilon_d<8d/\sigma^2$.*

Corollary 2. *If $n_d\neq 0$ then there are only finitely many d with $h(d)=1$.*

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