36. A Note on the Artin Map. II*)

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This is a continuation of my preceding paper [2] which will be referred to as (I) in this paper.[1] In (I), we defined, for a finite Galois extension K/k of number fields, a monoid homomorphism (a generalized Artin map)

$$\alpha_{K/k}: I(K/k) \longrightarrow C[G]_0, \ G=G(K/k),$$

where $I(K/k)$ denotes the monoid of nonzero integral ideals $\mathfrak{a}$ of k whose prime factors are all unramified in K and $C[G]_0$ denotes the center of the group ring $C[G]$. We, then, obtained a condition for the finiteness of the image of $\alpha_{K/k}$ in terms of characters (I. Theorem). In this paper, we shall study the kernel of $\alpha_{K/k}$ in a similar way. It will turn out that the structure of the kernel becomes simpler if the group G becomes away from being *abelian*.

§1. **Center of G.** Let G be a finite group. We shall denote by $\mathrm{Irr}(G)$ the set of all irreducible C-characters of G. For each $\chi \in \mathrm{Irr}(G)$, we put

$$\chi^*(x)=\frac{\chi(x)}{\chi(1)}, \quad x \in G.$$

As is well-known, we have $|\chi^*(x)| \leqq 1$ for all x, χ.[2] In this context, it is to be noted that

(1.1) $\qquad |\chi^*(x)|=1 \quad$ for all x, $\chi \Leftrightarrow G$ is abelian.

In this paper, we are interested in the following property (Z) of G which is weaker than (1.1):

(Z) There is an $x \neq 1$ in G such that $|\chi^*(x)|=1$ for all $\chi \in \mathrm{Irr}(G)$.

(1.2) **Proposition.** *G satisfies $(Z) \Leftrightarrow$ the center of G is nontrivial.*

Proof. For an $x \in G$, let $Z(x)$ be the centralizer of x. Our assertion follows from the following chains of equivalences: x is in the center of $G \Leftrightarrow G=Z(x) \Leftrightarrow [G]=[Z(x)]$[3] $\Leftrightarrow \sum_{\chi \in \mathrm{Irr}(G)} \chi(1)^2=[G]=[Z(x)]=\sum_{\chi \in \mathrm{Irr}(G)} |\chi(x)|^2 \Leftrightarrow |\chi(x)|=\chi(1)$ for all $\chi \Leftrightarrow |\chi^*(x)|=1$ for all χ. Q.E.D.

(1.3) **Remark.** Any nilpotent group $G(\neq 1)$ satisfies (Z). On the other hand, let $G=H\cdot\langle\tau\rangle$, a semidirect product of an abelian normal subgroup H of odd ($\geqq 3$) order and a cyclic subgroup $\langle\tau\rangle$ such that $\tau\sigma\tau^{-1}=\sigma^{-1}$, $\sigma \in H$, $\tau^2=1$. Then G does not satisfy (Z) as its center is trivial. Such a group G appears as the Galois group of K/Q where K is the Hilbert class field of a

*) To the memory of Michio Kuga.

1) For example, we mean by (I.2) the item (2) in (I).

2) As for elementary facts on characters, see first three chapters (pp. 1–46) of I. M. Isaacs, Character Theory of Finite Groups, Academic Press, New York-London, 1976.

3) We denote by $[S]$ the cardinality of a set S.