28. A Note on the Hilbert Irreducibility Theorem

The Irreducibility Theorem and the Strong Approximation Theorem*),**)

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Introduction. Let k be a finite algebraic number field. For any irreducible polynomial $f(t, x) \in k(t)[x]$, let $U_{f,k}$ denote the set consisting of all $s \in k$ such that $f(s, x)$ is defined and irreducible in $k[x]$. A subset of k of this form is called a *basic Hilbert subset* of k. Further, an intersection of a non-empty Zariski open subset of k and a finite number of basic Hilbert subsets of k is called a *Hilbert subset* of k.

In this paper, we obtain the following theorem:

Main theorem. *Let Ω be the set of all primes of a finite algebraic number field k, let $\mathfrak{q}$ be an element of Ω, and let S be a finite subset of $\Omega-\{\mathfrak{q}\}$ such that $\Omega-S-\{\mathfrak{q}\}$ contains only non-archimedean primes of k. We choose an element $\alpha_{\mathfrak{p}}$ of k for each $\mathfrak{p} \in S$. Then, for any positive number ε and for any Hilbert subset H of k, there exists an element $\alpha \in H$ such that*

$$\begin{cases} |\alpha-\alpha_{\mathfrak{p}}|_{\mathfrak{p}}<\varepsilon & \text{for any } \mathfrak{p} \in S, \\ |\alpha|_{\mathfrak{p}} \leq 1 & \text{for any } \mathfrak{p} \in \Omega-S-\{\mathfrak{q}\}. \end{cases}$$

Clearly, this theorem shows that the Hilbert irreducibility theorem and the strong approximation theorem for k are compatible. It is easy to reduce this theorem to the Hilbert irreducibility theorem if S contains only non-archimedean primes, but it seems nontrivial if S contains archimedean primes.

We prove the theorem by modifying an argument in S. Lang [1], VIII, §1.

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§1. Hilbert sets and rational points of algebraic curves. Let k be a finite algebraic number field, and let H be a Hilbert subset of k. Then, for some non-empty Zariski open subset O of k, we can write $O \cap H = O \cap (\bigcap_{i=1}^{m} U_{f_i,k})$, where $f_i(t, x)$ is an irreducible polynomial in $k(t)[x]$ and $U_{f_i,k}$ is the basic Hilbert subset corresponding to f_i. Here, by multiplying an element of $k[t]$ and changing O if necessary, we may assume $f_i(t, x) \in k[t, x]$.

Let $f(t, x)$ be one of the $f_i(t, x)$. Let $\overline{k(t)}$ be the algebraic closure of $k(t)$, and write $f(t, x)=a(t) \prod_{h=1}^{l}(x-\alpha_h)$ $(a(t) \in k[t],\ \alpha_h \in \overline{k(t)})$. Let $f(t, x)=$

*) Dedicated to Professor Ichiro SATAKE on his sixtieth birthday.

**) This result was obtained when the author was a member of the Sonderforschungsbereich 170, Geometrie und Analysis in Göttingen.