21. Asymptotic Behavior of the Solution for an Elliptic Boundary Value Problem with Exponential Nonlinearity

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§1. **Introduction and results.** We consider the elliptic eigenvalue problem

(1.1) $$-\Delta u=\lambda e^u \text{ (in } \Omega), \qquad u=0 \text{ (on } \partial\Omega)$$

for $u\in C^2(\Omega)\cap C^0(\bar{\Omega})$ and $\lambda\in \boldsymbol{R}_+\equiv(0,+\infty)$, where $\Omega\subset\boldsymbol{R}^2$ is a bounded domain with smooth boundary $\partial\Omega$. We say that $\kappa\in\Omega$ is a core of Ω if it is a critical point of $k(x)=K(x,x)$, where $K(x,y)=G(x,y)+(1/2\pi)\log|x-y|$, $G=G(x,y)$ being the Green function: $-\Delta G=\delta(x-y)$, $G|_{x\in\partial\Omega}=0$. When Ω is simply-connected, cores are finite. Furthermore, a core is unique if Ω is convex. For these facts, see Friedman [3] for example. On the other hand, for each core $\kappa\in\Omega$ satisfying a generic constrain, a branch S^* of the solutions $\{(u,\lambda)\}$ for (1.1) is constructed by the method of singular perturbation such a way that u makes one-point blow-up at κ as $\lambda\downarrow 0$. This fact has been established by Weston [9], Mosley [6] and Wente [8].

In the present note we show that conversely each family of solutions makes finite-point blow-up for star-shaped Ω as $\lambda\downarrow 0$, unless it approaches to the trivial solution $u=0$ of (1.1) for $\lambda=0$. More precisely,

Theorem. If Ω is simply connected and the family of solutions $\{u\}$ of (1.1) accumulates as $\lambda\downarrow 0$ to $v=8\pi E_\kappa(x)$ in $W^{1,p}(\Omega)$ $(1<p<2)$ and in $C(\bar{\Omega}\backslash\{\kappa\})$, then $\kappa\in\Omega$ is a core and the function $E_\kappa=E_\kappa(x)$ solves $-\Delta E_\kappa=\delta(\kappa)$ and $E_\kappa|_{\partial\Omega}=0$.

Spruck [7] has studied a similar property for Sinh-Gordon equation in the rectangular domain $R\subset\boldsymbol{R}^2$. We are much inspired by his work, but the finiteness of a blow-up point does not follow from his argument for general domains. Our result extends to other semilinear eigenvalue problem in two-dimensional domains with exponentially-dominated nonlinearities, and details will be published elsewhere.

§2. **Outline of Proof.** The proof is divided into three parts:

Claim 1. When Ω is star-shaped, then $\Sigma\equiv\lambda\int_\Omega e^u dx$ is bounded as $\lambda\downarrow 0$.

Claim 2. If $\{\Sigma\}$ is bounded, then $\{u\}$ accumulates to a $v\in W^{1,p}(\Omega)\cap C^\infty(\bar{\Omega}\backslash\{\kappa_1,\cdots,\kappa_l\})$ for some finite points $\kappa_1,\cdots,\kappa_l\in\Omega$.

Claim 3. If $\{\kappa_1,\cdots,\kappa_l\}=\{\kappa\}$, we have $\Sigma\to 8\pi$ and $v=8\pi E_\kappa$ with some

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