83. A Note on the Artin Map

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Let K/k be a finite Galois extension of algebraic number field with the Galois group $G=G(K/k)$, $\mathfrak{p}$ a prime ideal of k unramified for K/k and $\mathfrak{P}$ be a prime factor of $\mathfrak{p}$ in K. Denote by $\left[\dfrac{K/k}{\mathfrak{P}}\right]$ the Frobenius automorphism of $\mathfrak{P}$. For an element $\sigma\in G$, denote by $C(\sigma)$ the conjugate class containing σ, by $h(\sigma)$ the cardinality of $C(\sigma)$ and by $a(\sigma)$ the following element in the center $\boldsymbol{C}[G]_0$ of the group ring $\boldsymbol{C}[G]$:

$$a(\sigma)=\frac{1}{h(\sigma)}\sum_{\tau\in C(\sigma)}\tau. \tag{1}$$

For $\sigma=\left[\dfrac{K/k}{\mathfrak{P}}\right]$, we may write, without ambiguity, $C_{\mathfrak{p}}$, $h_{\mathfrak{p}}$, $a_{\mathfrak{p}}$, instead of $C(\sigma)$, $h(\sigma)$, $a(\sigma)$, respectively. One verifies easily that

$$a_{\mathfrak{p}}=\frac{1}{g_{\mathfrak{p}}}\sum_{\mathfrak{P}\mid\mathfrak{p}}\left[\frac{K/k}{\mathfrak{P}}\right]=\frac{1}{n}\sum_{\sigma\in G}\left[\frac{K/k}{\mathfrak{P}^{\sigma}}\right],\qquad n=[K:k], \tag{2}$$

where $g_{\mathfrak{p}}$ means the number of distinct prime factors of $\mathfrak{p}$ in K. We shall denote by $\alpha_{K/k}(\mathfrak{p})$ the element in $\boldsymbol{C}[G]_0$ defined by any member of the equalities (2). When K/k is abelian, $\alpha_{K/k}(\mathfrak{p})$ is an element of G and we have

$$\alpha_{K/k}(\mathfrak{p})=\left(\frac{K/k}{\mathfrak{p}}\right)\qquad\text{(Artin symbol).} \tag{3}$$

Back to any Galois extension K/k, put

$$I(K/k)=\{\mathfrak{a}\,;\ \text{ideal }(\neq 0)\text{ in }\mathfrak{o}_k,\ (\mathfrak{a},\Delta_{K/k})=1\}, \tag{4}$$

where $\mathfrak{o}_k$ is the ring of integers of k and $\Delta_{K/k}$ denotes the relative discriminant of K/k. If

$$\mathfrak{a}=\prod_{\mathfrak{p}}\mathfrak{p}^{\nu_{\mathfrak{p}}(\mathfrak{a})},\qquad \mathfrak{a}\in I(K/k), \tag{5}$$

is the factorization of $\mathfrak{a}$ in k, we put

$$\alpha_{K/k}(\mathfrak{a})=\prod_{\mathfrak{p}}\alpha_{K/k}(\mathfrak{p})^{\nu_{\mathfrak{p}}(\mathfrak{a})}. \tag{6}$$

The map $\alpha_{K/k}$ whose domain of definition is now $I(K/k)$ is, as is easily seen, a homomorphism of the multiplicative semigroup $I(K/k)$ into the multiplicative semigroup of the commutative ring $\boldsymbol{C}[G]_0$ sending the identity $\mathfrak{o}_k$ to the identity 1_G. When K/k is abelian, the image of $\alpha_{K/k}$ is just the group G (by the density theorem due to Tschebotareff) and the determination of fibres of $\alpha_{K/k}$ is the content of the Artin reciprocity in class field theory. Therefore it is natural to study the image and fibres of the map $\alpha_{K/k}: I(K/k)\to\boldsymbol{C}[G]_0$ for nonabelian Galois extension K/k. Since the cardinality of the image of $\alpha_{K/k}$ is the order of G when K/k is abelian, let us start our study of $\alpha_{K/k}$ with a criterion for the finiteness of the image. To do this, we need some