77. The Selberg Zeta Function and the Determinant of the Laplacians

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1. Outline. Let G be a noncompact connected semi-simple Lie group of rank 1 with finite center, and Γ its cofinite discrete subgroup. For such pairs of G and Γ, the Selberg theory is constructed. If we put K to be a maximal compact subgroup of G, then $M := \Gamma\backslash G/K$ is a Riemannian manifold and the Laplacian Δ over $L^2(M)$ is defined. For compact M, S. Minakshisundaram and Å. Pleijel prove the regularity of the spectral zeta function of Δ at the origin, by which we can define the determinant of Δ. When Γ is torsion-free and cocompact, A. Voros and P. Sarnak show that the Selberg zeta function with local factor is expressed as the determinant of Δ and calculate the local factor explicitly. We will generalize this type of determinant expression for more general G and Γ. Generally, torsion subgroups of Γ cause new local factors of the Selberg zeta function. Moreover, when Γ is not cocompact, that is, M is not compact, the continuous spectrum appears and contributes to the determinant of Δ. In the following sections, we generalize the theorem of Minakshisundaram-Pleijel to some noncompact cases, and give explicit forms of all the local factors of the Selberg zeta function and the contribution of the continuous spectrum.

2. The general program. Let $0=\lambda_0<\lambda_1\leq\lambda_2\leq\cdots$ be the eigen values of Δ. We induce the spectral zeta function generalized by a real variable $s>2\rho_0$; $\zeta(w,s,\Delta) := \sum_{n=0}^{\infty}(\lambda_n-s(2\rho_0-s))^{-w}$, for the purpose of expressing the Selberg zeta function. Here ρ_0 is the constant depending on G, which is defined in [1, p. 4]. For examining poles of $\zeta(w,s,\Delta)$, we use the trace formula of Selberg in the form of general case (Gangolli-Warner [1]). Taking the test function $h(r^2+\rho_0^2) := \exp(-(r^2+(s-\rho_0)^2)t)$ $(t>0)$, the trace formula has the form

$$(1)\qquad \sum_{n=0}^{\infty}\exp(-(\lambda_n-s(2\rho_0-s))t)=I(t)+E(t)+H(t)+P(t)-\mathrm{Tr}_C(t)$$

whose right side has the terms of identity, elliptic, hyperbolic and parabolic conjugacy classes, and the removed trace of the continuous spectrum. The Mellin transformation shows that the behavior of (1) as $t\to0$ determines the poles of $\zeta(w,s,\Delta)$. Indeed, the behavior t^a (resp. $t^a\log t$) causes the simple (resp. double) pole at $w=-a$ of the function $\Gamma(w)\zeta(w,s,\Delta)$. Studying each term in (1), the main difficulty is the treatment of Tr_C. It has the form

$$(2)\qquad \mathrm{Tr}_C(t) := -(4\pi)^{-1}\int_{-\infty}^{\infty}h(r^2+\rho_0^2)(\varphi'/\varphi)(\rho_0+ir)dr+4^{-1}\lim_{s\to\rho_0}\mathrm{tr}\,\Phi(s)h(\rho_0^2),$$