75. A Note on the Mean Value of the Zeta and L-functions. VI

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Continuing our previous work [3] we show an explicit formula for

$$I_4(T,\Delta)=(\Delta\sqrt{\pi})^{-1}\int_{-\infty}^{\infty}\left|\zeta\left(\frac{1}{2}+i(T+t)\right)\right|^4 e^{-(t/\Delta)^2}dt.$$

We retain the notations introduced in [3] from the theory of automorphic functions.

We define $D_\Delta(r,s)$ as the analytic continuation of

$$\int_0^\infty x^{r-1}(1+x)^{-s}\exp\left(-\left(\frac{\Delta}{2}\log(1+x)\right)^2\right)dx,$$

and put

$$\begin{aligned}\Psi(u,v,w,z;\xi)=&-i(2\pi)^{z-v-2}\cos\left(\frac{\pi}{2}(u-w)\right)\\ &\times\int_{-\infty i}^{\infty i}\sin\left(\frac{\pi}{2}(u+v+w+z-2r)\right)\Gamma\left(\frac{1}{2}(u+v+w+z-1)+\xi-r\right)\\ &\times\Gamma\left(\frac{1}{2}(u+v+w+z-1)-\xi-r\right)\Gamma(1-w-z+r)\Gamma(1-u-z+r)D_\Delta(r,z)dr.\end{aligned}$$

The path of integration is curved to ensure that the poles of the first three factors on the integrand lie on the right of the path and those of the remaining factors on the left; it is assumed that u, v, w, z, ξ are such that the contour can be drawn. Also we define $\Phi(u,v,w,z;\xi)$ to be the one which is obtained by replacing in the above the factors $\cos((\pi/2)(u-w))$ and $\sin((\pi/2)(u+v+w+z-2r))$ by $\cos(\pi\xi)$ and $\cos((\pi/2)(u+w+2z-2r))$, respectively. It can be shown that Ψ and Φ admit meromorphic continuations over the entire $\boldsymbol{C}^5$; hereafter the symbols Ψ and Φ will denote these meromorphic functions. Ψ and Φ are of rapid decay: Uniformly for any bounded u, v, w, z and for any fixed $c>0$ we have $\Psi=O(|\xi|^{-c}e^{-\pi|\xi|})$ and $\Phi=O(|\xi|^{-c})$ when $|\mathrm{Im}\,\xi|$ tends to infinity while $\mathrm{Re}\,\xi$ remains bounded. Further we put, with an obvious abuse of notation,

$$\Theta(\xi;T,\Delta)=2\,\mathrm{Re}\,\{(\Psi-\Phi)(P_T;i\xi)\},$$
$$\Lambda(k;T,\Delta)=2\,\mathrm{Re}\left\{\Psi\left(P_T;k-\frac{1}{2}\right)\right\},$$

where P_T is the point $(\frac{1}{2}+iT,\frac{1}{2}-iT,\frac{1}{2}+iT,\frac{1}{2}-iT)$, $T>0$, and $k=1,2,3\cdots$. We should note that for any fixed $B>0$ $\Lambda(k;T,\Delta)=O(\Delta^{-k})$ if $k\leq B$, and $=O((k\Delta)^{-B})$ if $k>B$, where the implied constants depend only on B.

Then our main result is as follows: