60. On Affine Surfaces whose Cubic Forms are Parallel Relative to the Affine Metric[†]

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Let M^n be a nondegenerate affine hypersurface in affine space $\boldsymbol{R}^{n+1}$ and denote by ∇, h and $\hat{\nabla}$ the induced connection, the affine metric, and the Levi-Civita connection for h, respectively. (We follow the terminology of [4].) Let $C=\nabla h$ be the cubic form.

It is a classical theorem that if $C=0$, then M^n is a quadratic hypersurface. In [5], it is shown that for $n=2$ the condition $\nabla C=0$, $C\neq 0$ characterizes, up to an equiaffine congruence, a Cayley surface, namely, the graph of the cubic polynomial $z=xy-y^3/3$. For an arbitrary dimension, [1] has shown that the tensor ∇C is totally symmetric (i.e. symmetric in all its indices) if and only if $\hat{\nabla} C$ is totally symmetric, and this symmetry condition implies that M^n is an affine hypersphere. It is also shown that the condition $\nabla C=0$, $C\neq 0$ implies that M^n is an improper affine hypersphere such that h is hyperbolic metric and the Pick invariant J is 0. As for the case $n=2$, affine spheres M^2 whose affine metric h is flat have been completely determined in [3], although the case where h is elliptic was already done in [2].

In this note, we study affine surfaces with the property $\hat{\nabla} C=0$, $C\neq 0$, and prove the following classification.

Theorem. *If a nondegenerate affine surface in $\boldsymbol{R}^3$ satisfies $\hat{\nabla} C=0$, $C\neq 0$, then it is equiaffinely congruent to a piece of one of the following surfaces:*

1) *the graph of $z=1/xy$ (h: elliptic);*
2) *the graph of $z=1/(x^2+y^2)$ (h: hyperbolic and $J\neq 0$);*
3) *Cayley surface (h: hyperbolic and $J=0$).*

The proof is given along the following lines. First, from the results quoted from [1] we see that the surface is an affine sphere. Next, we show that the assumption of the theorem implies that the connection $\hat{\nabla}$ is flat by using the argument similar to that in [5]. Now the result in [3] leads to our classification by using a concrete procedure to show that the graph of $z=xy+\varphi(y)$, where φ is an arbitrary cubic polynomial, is equiaffinely congruent to the Cayley surface.

Proof of the theorem. *Step* 1. We show that $\hat{\nabla} C=0$ implies that M^2 is an affine sphere. Indeed, from [1] we know that ∇C is totally symmetric, and this implies our assertion.

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