56. *Inverse Map Theorem in the Ultra-F-differentiable Class*

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Let M_p, $p=0, 1, 2, \cdots$, be a sequence of positive numbers. We assume that $M_0=1$ and that there is a constant $H \geqq 1$ such that

(1) $$(M_p/p\,!)^{1/p} \leqq H(M_q/q\,!)^{1/q}, \qquad \text{if } 1 \leqq p \leqq q.$$

Let X, Y be Banach spaces and $U \subset X$ be an open set. A C^∞-map (in the sense of Frechét derivatives) $f: U \to Y$ is said to belong to the ultra-F-differentiable class $\alpha\{M_p\}$, if there are constants $C \geqq 0$ and $h>0$ such that, if $x \in U$ and $p \geqq 0$, then $\|f^{(p)}(x)\| \leqq Ch^p M_p$.

The main purpose of this note is to give an inverse map theorem (Theorem 2) in the class $\alpha\{M_p\}$ which is an improvement and a generalization of a similar theorem by H. Komatsu [2] in the following sense. Our result improves that of [2] in the sense that the condition (1) is simpler and a little weaker than the corresponding one in [2]

(2) $$(M_p/p\,!)^{1/(p-1)} \leqq H(M_q/q\,!)^{1/(q-1)}, \qquad \text{if } 2 \leqq p \leqq q.$$

Also, our theorem generalizes to the infinite dimensional case that of [2] which deals with the finite dimensional case. In order to prove his inverse map theorem Komatsu used majorant series. Essentially the same idea is used in this note, too. But in order to deal with the infinite dimensional case in a clear-cut way we resort to a convenient tool that was not used in [2]. It is the higher order chain rule of Faa' di Bruno [1], which reads, in the one dimensional case,

(3) $$(f \circ g)^{(p)}(x) = p\,! \sum_{j=1}^{p} f^{(j)}(g(x)) \sum_{|q|=j\ \|q\|=p} \prod_{i=1}^{p-j+1} \frac{1}{q_i\,!} \left(\frac{g^{(i)}(x)}{i\,!}\right)^{q_i},$$

where f and g are real-valued functions of a real variable, $q=(q_1, \cdots, q_{p-j+1})$ is a multi-index and

$$|q| = q_1 + q_2 + \cdots + q_{p-j+1}, \qquad \|q\| = q_1 + 2q_2 + \cdots + (p-j+1)q_{p-j+1}.$$

Let us first generalize the above rule to the infinite dimensional case. In the following theorem the symbol sym denotes the symmetrization of a multi-linear operator. For its definition and other notational conventions with respect to multi-linear operators see [3].

Theorem 1. *Let X, Y, Z be normed spaces and $U \subset X$, $V \subset Y$ be open. Let $g: U \to V$ and $f: V \to Z$ be C^p-maps. Then for $x \in U$*

(4) $$(f \circ g)^{(p)}(x) = \mathrm{sym}\left(p\,! \sum_{j=1}^{p} f^{(j)}(g(x)) \sum_{|q|=j\ \|q\|=p} \prod_{i=1}^{p-j+1} \frac{1}{q_i\,!} \left\{\frac{g^{(i)}(x)}{i\,!}\right\}^{q_i}\right).$$

Proof. Let x be an arbitrary fixed point of U. Put $y=g(x)$. For $h \in X$ with small norm we have, by Taylor's rule,