# 53. *Loop Groups and Related Affine Lie Algebras*

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(Communicated by Shokichi IYANAGA, M. J. A., June 13, 1989)

**Introduction.** We are concerned with the Lie group $\tilde{G}_k$ of $C^k$-loops in a connected, simply connected complex simple Lie group $G$, its Lie algebra $\tilde{\mathfrak{g}}_k$, and a central extension $\hat{\mathfrak{g}}_k$ of $\tilde{\mathfrak{g}}_k$. The Lie algebra $\tilde{\mathfrak{g}}$ of algebraic loops in the Lie algebra $\mathfrak{g}$ of $G$ has a universal central extension $\hat{\mathfrak{g}}$ called an affine Lie algebra, and the corresponding 2-cocycle $Z(\cdot, \cdot)$ was explicitly given in [1]. We extend the 2-cocycle of $\tilde{\mathfrak{g}}_k$ after [2], and get a central extension $\hat{\mathfrak{g}}_k$ of $\tilde{\mathfrak{g}}_k$. $\hat{\mathfrak{g}}$ is one of the simplest infinite-dimensional Kac-Moody algebras. The corresponding Kac-Moody group $\hat{G}$ is a 1-dimensional central extension of the group $\tilde{G}$ of algebraic loops in $G$ (cf. [1], [7], [4], and [5]).

Since the kernel of the adjoint action Ad of $\hat{G}$ on $\hat{\mathfrak{g}}$, is precisely the center $C$ of $\hat{G}$, $\tilde{G}\simeq\hat{G}/C$ acts on $\hat{\mathfrak{g}}$ through Ad, and the set of invariants in $\hat{\mathfrak{g}}$ under this action is just the center of $\hat{\mathfrak{g}}$. The action on $\hat{\mathfrak{g}}$ induces the adjoint action of $\tilde{G}$ on $\tilde{\mathfrak{g}}$. The main purpose of this article is to construct a completed version of this fact for the pair of the infinite-dimensional Lie group $\tilde{G}_k$ and the Lie algebra $\hat{\mathfrak{g}}_k$.

**§1. The coefficient extension from $C$ to $L_k=C^k(S^1)$.** Let $L_k:=C^k(S^1)$, the algebra of $C^k$-functions on $S^1$. This becomes a Banach algebra if we introduce a norm $|\cdot|_k$ as

$$|a|_k:=\sup_{r\in\boldsymbol{R},\, j=0,\cdots,k}|(\partial^j a)(e^{2\pi\sqrt{-1}r})| \qquad \text{for } a\in L_k,$$

where $\partial$ is a differential operator on $S^1$, defined by

$$(\partial a)(e^{2\pi\sqrt{-1}r}):=\frac{1}{2\pi\sqrt{-1}}\frac{d}{dr}a(e^{2\pi\sqrt{-1}r}).$$

Let $n$ be a positive integer and $i=0, 1, 2, \cdots, n$. Define derivations $D_i$ on the polynomial ring $P_{k;n}:=L_k[X_1, \cdots, X_n]$ by $D_iX_{i'}=\delta_{ii'}$, and $D_ia=0$ for $i'=1, 2, \cdots, n$, $a\in L_k$. For a bounded closed subset $B$ in $(L_k)^n$ and a non-negative integer $j$, put

$$|f|_{k;B,j}:=\sup_{m,b}|(D^m f)(b)|_k \qquad \text{for } f\in P_{k;n},$$

where sup is taken over all $m=(m_1, \cdots, m_n)\in(\boldsymbol{Z}_{\geq 0})^n$ satisfying $|m|:=m_1+\cdots+m_n\leqq j$, and all $b=(b_1, \cdots, b_n)\in B$, and $D^m$ means $D_1^{m_1}D_2^{m_2}\cdots D_n^{m_n}$. Let $C^{k;j}(B)$ be the completion of the normed space $(P_{k;n}, |\cdot|_{k;B,j})$.

Let $U$ be an open set in $(L_k)^n$, and $C^{k;j}(U)$ the space of maps $f: U\to L_k$ which satisfy that, for any $u\in U$, there exist a bounded closed neighbourhood $B$ of $u$ in $(L_k)^n$ and $g\in C^{k;j}(B)$ such that $f(b)=g(b)$ for $\forall b\in B$. We define $D^m f(u)$ as $D^m g(u)$ for $m\in(\boldsymbol{Z}_{\geq 0})^n$, $|m|\leqq j$.