6. On the Inequalities of Erdös-Turán and Berry-Esseen. II

By Petko D. PROINOV
Department of Mathematics, University of Plovdiv, Bulgaria

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This is continued from [1].

5. The ideas of the proofs of the results given in Sections 3 and 4 are similar. Here we shall prove only Theorem 1. The proof is based on some ideas of Sendov [3] and the author [2]. We begin with a well known lemma of Sendov, which he used in the approximation theory.

Lemma 1 ([3], [4]). *Let f be a periodic function with period 1, and let μ be its modulus of nonmonotonicity (on $\boldsymbol{R}$). Suppose also that $x \in \boldsymbol{R}$ and $\delta \geqq 0$. Then:*

(a) *The inequality $f(t) \leqq f(x)+\mu(2\delta)$ holds either for all $t \in [x, x+\delta]$, or for all $t \in [x-\delta, x]$.*

(b) *The inequality $f(t) \geqq f(x)-\mu(2\delta)$ holds either for all $t \in [x, x+\delta]$, or for all $t \in [x-\delta, x]$.*

In what follows, a periodic function K with period 1 is said to be a *kernel* if it is nonnegative, even and $\int_0^1 K(t)dt=1$.

Lemma 2. *Let f be as in Theorem 1, and let μ be its modulus of nonmonotonicity. Suppose also that K is a kernel, and set*

$$\mathcal{K}(f\,; x)=\int_0^1 f(t)K(t-x)dt \qquad \text{for all } x \in \boldsymbol{R}.$$

Then:

(i) *For every $\delta \in [0, 1/2]$,*

$$\|f\| \leqq \mu(4\delta)+\|\mathcal{K}(f, \cdot)\|+2(2\|f\|-\mu(4\delta))\int_\delta^{1/2} K(t)dt.$$

(ii) *For every $\delta \geqq 1/2$,*

$$\|f\| \leqq \mu(4\delta)+\|\mathcal{K}(f\,; \cdot)\|.$$

Proof. (i) Let $\delta \in [0, 1/2]$ and $x \in \boldsymbol{R}$. First we shall prove that

$$|f(x)|\int_{-\delta}^{\delta} K(t)dt \leqq \mu(4\delta)\int_{-\delta}^{\delta} K(t)dt+2\|f\|\int_\delta^{1/2} K(t)dt+\|\mathcal{K}(f\,; \cdot)\|. \tag{1}$$

According to Lemma 1-(a) the inequality

$$f(t) \leqq f(x)+\mu(4\delta) \tag{2}$$

holds either for all $t \in [x, x+2\delta]$, or for all $t \in [x-2\delta, x]$.

Suppose first that (2) holds for all $t \in [x, x+2\delta]$. In this case we shall obtain an upper bound for the value of $\mathcal{K}(f\,; x+\delta)$. We have

$$\mathcal{K}(f\,; x+\delta)=\int_{-1/2}^{1/2} f(t+x+\delta)K(t)dt \tag{3}$$

since f is a periodic function with period 1. Now we write $\mathcal{K}(f\,; x+\delta)$ in the form