# 39. *Zeta Zeros, Hurwitz Zeta Functions and* $L(1,\chi)$

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§1. **Introduction.** Let $a$ be a positive number $<1$. We are concerned with the value distribution of the Hurwitz zeta function $\zeta(s,a)=\sum_{n=0}^{\infty}\frac{1}{(n+a)^s}$ (for $\mathrm{Re}(s)>1$), at the zeros of the Riemann zeta function $\zeta(s)$.

Although $\zeta(s,a)$ has many good properties like $\zeta(s)$, it fails to have the Euler product formula except when $a=1/2$, in which case we have $\zeta(s,1/2)=(2^s-1)\zeta(s)$. So it might be interesting to clarify how any problem concerning $\zeta(s,a)$ depends on $a$. We assume the Riemann Hypothesis throughout this article and prove the following theorem. To state our theorem, we put $L_a(1)=\sum_{n=1}^{\infty}\frac{e(-na)}{n}$ with $e(y)=e^{2\pi iy}$ and $\Lambda(x)=\log p$ if $x=p^k$ with a prime number $p$ and an integer $k\geq 1$, and $=0$ otherwise. We denote the imaginary parts of the zeros of $\zeta(s)$ by $\gamma$.

**Theorem.** *For any positive* $a<1$,

$$\lim_{T\to\infty}\frac{2\pi}{T}\sum_{0<\gamma\leq T}\zeta\left(\frac{1}{2}+i\gamma,a\right)=-\Lambda\left(\frac{1}{a}\right)-L_a(1).$$

From this theorem we see first that for any integer $k\geq 2$,

$$\begin{aligned}1&+\frac{1}{2}+\cdots+\frac{1}{k-1}-\frac{k-1}{k}+\frac{1}{k+1}+\frac{1}{k+2}+\cdots\\&+\frac{1}{2k-1}-\frac{k-1}{2k}+\frac{1}{2k+1}+\cdots\\&=\log k,\end{aligned}$$

since $\sum_{b=1}^{k-1}\zeta(s,b/k)=(k^s-1)\zeta(s)$ and $\sum_{b=1}^{k-1}\Lambda(k/b)=\sum_{m|k}\Lambda(m)=\log k$. (We know, of course, that this can be proved in an elementary way.)

We see next that for any primitive character $\chi \bmod q\geq 3$,

$$\begin{aligned}\lim_{T\to\infty}\frac{2\pi}{T}\sum_{b=1}^{q-1}\bar{\chi}(b)\sum_{0<\gamma\leq T}\zeta\left(\frac{1}{2}+i\gamma,\frac{b}{q}\right)&=-\sum_{b=1}^{q-1}\bar{\chi}(b)\Lambda\left(\frac{q}{b}\right)-\sum_{b=1}^{q-1}L_{b/q}(1)\bar{\chi}(b)\\&=-\Lambda(q)-\sum_{n=1}^{\infty}\frac{1}{n}\sum_{b=1}^{q-1}e\left(-\frac{b}{q}n\right)\bar{\chi}(b)\\&=-\Lambda(q)-\bar{\tau}(\chi)L(1,\chi),\end{aligned}$$

where $L(s,\chi)$ is the Dirichlet $L$-function and $\tau(\chi)=\sum_{b=1}^{q}\chi(b)e(b/q)$. Moreover since $\zeta(s,b/q)$ can be written as a linear combination of $L$-functions, we get the following new expressions of $L(1,\chi)$ (cf. also [5] and [6] for other type of expressions).

**Corollary.** *For any primitive character* $\chi \bmod q\geq 3$,