35. An Elementary Proof of an Order Preserving Inequality

By Takayuki FURUTA
Department of Mathematics, Faculty of Science, Hirosaki University

(Communicated by Kôsaku YOSIDA, M. J. A., May 12, 1989)

An operator means a bounded linear operator on a Hilbert space. By only using the idea of polar decomposition, here we give an elementary proof of the following "order preserving inequality" in [1].

Theorem. *If $A \geqq B \geqq 0$, then for each $r \geqq 0$*

(1) $$(B^r A^p B^r)^{1/q} \geqq B^{(p+2r)/q}$$

holds for each p and q such that $p \geqq 0$, $q \geqq 1$ and $(1+2r)q \geqq p+2r$.

Proof. First of all, we cite (*) by Löwner-Heinz theorem.

(*) $A \geqq B \geqq 0$ ensures $A^\alpha \geqq B^\alpha$ for any $\alpha \in [0,1]$.

In the case $1 \geqq p \geqq 0$, the result is obvious by (*). We have only to consider $p \geqq 1$ and $q=(p+2r)/(1+2r)$ since (1) for values q larger than $(p+2r)/(1+2r)$ follows by (*). We may assume that A and B are invertible without loss of generality. Let $B^r A^{p/2} = UH$ be the polar decomposition of the invertible operator $B^r A^{p/2}$ where U means the unitary and $H = |B^r A^{p/2}|$. In the case $1 \geqq 2r \geqq 0$, $A^{2r} \geqq B^{2r}$ holds by (*), then for $q=(p+2r)/(1+2r)$

$$\begin{aligned} B^{-r}(B^r A^p B^r)^{1/q} B^{-r} &= B^{-r}(UH^2U^*)^{1/q} B^{-r} = B^{-r} U H^{2/q} U^* B^{-r} \\ &= A^{p/2} H^{-1} H^{2/q} H^{-1} A^{p/2} = A^{p/2} (H^2)^{1/q-1} A^{p/2} \\ &= A^{p/2} (A^{-p/2} B^{-2r} A^{-p/2})^{1-1/q} A^{p/2} \\ &\geqq A^{p/2} (A^{-p/2} A^{-2r} A^{-p/2})^{(p-1)/(p+2r)} A^{p/2} \\ &= A \geqq B, \end{aligned}$$

so we have the following (2) for $q=(p+2r)/(1+2r)$ and for any $r \in [0, 1/2]$

(2) $$(B^r A^p B^r)^{1/q} \geqq B^{1+2r}.$$

Put $A_1 = (B^r A^p B^r)^{1/q}$ and $B_1 = B^{1+2r}$. Repeating (2) again for $A_1 \geqq B_1 \geqq 0$, $0 \leqq r_1 \leqq 1/2$ and $p_1 \geqq 1$

$$(B_1^{r_1} A_1^{p_1} B_1^{r_1})^{1/q_1} \geqq B_1^{1+2r_1} \qquad \text{for } q_1 = (p_1+2r_1)/(1+2r_1).$$

Put $p_1 = q \geqq 1$ and $r_1 = 1/2$, then

(3) $$\{B^{2r+1/2} A^p B^{2r+1/2}\}^{1/q_1} \geqq B^{2(1+2r)}.$$

Put $s=2r+1/2$. Then $q_1=(p_1+2r_1)/(1+2r_1)=(p+2s)/(1+2s)$ since $p_1=q$ and $2(1+2r)=1+2s$. Consequently (3) means that (2) holds for $r \in [0, 3/2]$ since $r \in [0, 1/2]$ and $s=2r+1/2$ and repeating this method, (2) holds for each $r \geqq 0$, that is, (1) is shown.

Reference

[1] T. Furuta: $A \geqq B \geqq 0$ assures $(B^r A^p B^r)^{1/q} \geqq B^{(p+2r)/q}$ for $r \geqq 0$, $p \geqq 0$, $q \geqq 1$ with $(1+2r)q \geqq p+2r$. Proc. Amer. Math. Soc., **101**, 85–88 (1987).