26. Notes on Quasi-polarized Varieties

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0. Let V be a variety, which means, an irreducible reduced projective scheme over an algebraically closed field $\mathfrak{K}$ of any characteristic. A line bundle L on V is said to be *nef* if $LC \geqq 0$ for any curve C in V. It is said to be *big* if $\kappa(L)=n=\dim V$. In case L is nef, it is big if and only if $L^n>0$ (cf. [2; (6.5)]). When L is nef and big, the pair (V, L) will be called a *quasi-polarized variety*. In this note we report several generalizations of results on polarized manifolds. For details see [4].

1. We have $\chi(V, tL)=\sum_{j=0}^{n} \chi_j t^{[j]}/j!$ for some integers $\chi_0, \chi_1, \cdots, \chi_n$ where $t^{[j]}=t(t+1)\cdots(t+j-1)$ and $t^{[0]}=1$. By the Riemann-Roch theorem we have $\chi_n=L^n$. Moreover, if V is normal, we have

$$-2\chi_{n-1}=(\omega+(n-1)L)L^{n-1}$$

for the canonical divisor ω of V. We set $g(V, L)=1-\chi_{n-1}$, which is called the *sectional genus* of (V, L). We set $\Delta(V, L)=n+L^n-h^0(V, L)$, which is called the *$\Delta$-genus* of (V, L). We conjecture:

Both the Δ-genus and the sectional genus are non-negative for any quasi-polarized variety. Moreover, $\Delta=0$ if and only if $g=0$.

We expect further that we can classify somehow (V, L)'s with small Δ and g.

2. First of all we have the following

Theorem. *$\Delta(V, L) \geqq 0$ for any quasi-polarized variety. Moreover, if $\Delta=0$, there are a polarized variety (W, H) and a birational morphism $f: V \to W$ such that $L=f^*H$ and $\Delta(W, H)=0$.*

We have a complete classification of polarized varieties of Δ-genus zero (cf. [1]). In particular $g(W, H)=0$ and H is very ample. Hence $g(V, L)=0$ and $\mathrm{Bs}|L|=\varnothing$ if $\Delta(V, L)=0$.

3. From now on, we assume $\mathrm{char}(\mathfrak{K})=0$, since we need vanishing theorems of Kodaira-Kawamata-Viehweg type. Using the above theorem we obtain the following

Theorem. *Let (V, L) be a normal quasi-polarized variety with $\dim V=n$. Suppose that $h^n(V, -tL)=0$ for any t such that $0<t \leqq n$. Then there is a birational morphism $f: V \to \boldsymbol{P}^n$ such that $L=f^*\mathcal{O}(1)$.*

4. Next we improve results in [3]. An element of $\mathrm{Pic}(V) \otimes \boldsymbol{Q}$ is called a *$\boldsymbol{Q}$-bundle* on V. We define $\boldsymbol{Q}$-valued intersection numbers of $\boldsymbol{Q}$-bundles and the nefness of them in the natural way.

Let $\pi: M \to V$ be a desingularization of a normal variety V and set $S=\{x \in V \mid \dim \pi^{-1}(x)>0\}$ and $E=\pi^{-1}(S)$. Then π is said to be nice if E is a