24. On Some Inequalities in the Theory of Uniform Distribution. I

By Petko D. PROINOV and Nedyalka A. MITREVA
Department of Mathematics, University of Plovdiv, Bulgaria

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In this note, we present two inequalities for the supremum norm and the oscillation of a function satisfying a one-sided Lipschitz condition on the interval $E=[0,1]$ and having equal values at the end points. As special cases of them we obtain two estimates for the φ-discrepancy of a sequence of real numbers, with respect to a distribution function satisfying a Lipschitz condition on E. The results generalize some inequalities of LeVeque [3], Yurinskii [12], Niederreiter ([5], [6]), and Proinov ([7], [8]).

1. Definition 1. A real-valued function f is said to satisfy the *right Lipschitz condition* on E with a positive constant L if

$$f(x)-f(y)\leqq L(x-y) \qquad \text{for } x, y \in E \text{ with } x>y. \tag{1}$$

Analogously, f is said to satisfy the left Lipschitz condition if

$$f(x)-f(y)\geqq -L(x-y) \qquad \text{for } x, y \in E \text{ with } x>y. \tag{2}$$

The function f is said to satisfy the *one-sided Lipschitz condition* on E with constant L if either (1) or (2) holds.

It is easy to prove that if a function satisfies a one-sided Lipschitz condition on E, then it is a function of bounded variation on E. For a bounded function f on E, we denote by $\|f\|$ and $[f]$ its supremum norm and its oscillation, respectively.

Theorem 1. *Let a function f satisfy the one-sided Lipschitz condition on E with constant L, and let $f(0)=f(1)$ and $\|f\|\leqq L$. Then for any nondecreasing nonnegative function φ on $[0, \infty)$,*

$$F(\|f\|)\leqq L\int_0^1 \varphi(|f(x)|)dx \tag{3}$$

and

$$2F\left(\frac{1}{2}[f]\right)\leqq L\int_0^1 \varphi(|f(x)|)dx, \tag{4}$$

where the function F is defined on $[0, \infty)$ by

$$F(x)=\int_0^x \varphi(t)dt. \tag{5}$$

Proof. We shall prove only (4) since (3) can similarly be proved. We may assume that f satisfies a left Lipschitz condition since the other case follows immediately from this one (replacing f by $-f$). Now we extend f on $\boldsymbol{R}$ with period 1. Then it is easy to prove that the extended function f satisfies the left Lipschitz condition on the whole real line $\boldsymbol{R}$ with constant L. First we shall prove that the inequality