23. A Problem on Quadratic Fields

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Let k be a quadratic field, Δ_k the discriminant and M_k the Minkowski constant:

$$M_k=\begin{cases}\frac{1}{2}\sqrt{\Delta_k} & \text{if } k \text{ is real,}\\ \frac{2}{\pi}\sqrt{-\Delta_k} & \text{if } k \text{ is imaginary.}\end{cases}$$

Consider the finite set of prime numbers

$$\Pi_k=\{p, \text{ rational prime}\,;\ p\leqq M_k\}.$$

There are exactly 8 fields for which $\Pi_k=\phi$. They make up an exceptional family

$$E_8=\{k=\boldsymbol{Q}(\sqrt{m})\,;\ m=-1,\ \pm 2,\ \pm 3,\ 5,\ -7,\ 13\}.$$

For any k, let χ_k denote the Kronecker character. The character splits Π_k into 3 disjoint parts:

$$\Pi_k^0=\{p\in\Pi_k\,;\ \chi_k(p)=0\},$$
$$\Pi_k^-=\{p\in\Pi_k\,;\ \chi_k(p)=-1\},$$
$$\Pi_k^+=\{p\in\Pi_k\,;\ \chi_k(p)=+1\}.$$

Consider, next, the 3 families of fields:

$$K^0=\{k\,;\ \Pi_k=\Pi_k^0\},$$
$$K^-=\{k\,;\ \Pi_k=\Pi_k^-\},$$
$$K^+=\{k\,;\ \Pi_k=\Pi_k^+\}.$$

The problem is to determine explicitly the 3 families. Since E_8 is common to all 3 families, it is enough to determine K^0-E_8, K^--E_8, K^+-E_8, respectively.

(I) K^0-E_8. This is the easiest part of the problem and one settles it completely. Namely,

(1) $$K^0-E_8=\{k=\boldsymbol{Q}(\sqrt{m})\,;\ m=-5,\ \pm 6, 7, 10, 15, \pm 30\}.$$

Proof of (1). Let p_n denote the n th prime. Using the well-known Chebyshev's inequality, $p_{n+1}<2p_n$, $n\geqq 1$, one proves by induction that

(2) $$p_{n+1}^2<\frac{1}{4}p_1p_2\cdots p_n \qquad \text{when } n\geqq 5.$$

For any $k\in K^0-E_8$, choose n so that $p_n\leqq M_k<p_{n+1}$. Since $p_n\leqq M_k$, we have $p_1\cdots p_n\,|\,\Delta_k$ by definition of K^0, and so

$$p_{n+1}^2>M_k^2\geqq\frac{1}{4}\,|\Delta_k|\geqq\frac{1}{4}p_1\cdots p_n.$$

Hence, by (2), $n\leqq 4$ and $M_k<p_5=11$, from which one easily verifies (1).

(II) K^--E_8. This part of the problem is almost settled by H. M. Stark