# 13. *On Pathwise Projective Invariance of Brownian Motion.* I[†),*)]

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**Introduction.** Brownian motion with parameter in Riemannian space was introduced by P. Lévy [3]. He also considered white noise representation of Brownian motion in connection with geometric structure of its parameter space. In line with his idea we start with the simplest case of usual 1-parameter Brownian motion. The parameter space is considered the projective space $\boldsymbol{P}^1$ rather than $\boldsymbol{R}^1$.

In part I, we study an invariance property of the path space. This property is a reflection of the projective structure of $\boldsymbol{P}^1$. We also see that this invariance characterizes the Brownian motion between 1-parameter self-similar Gaussian processes.

In part II, the type of the group action which describes the above invariance will be determined as a *discrete series representation of index 2* in term of the theory of unitary representation.

In part III, we will consider a generalization of the partially invariance in § 3. Proposition 4 will be extended to multi-parameter case. The Möbius group will appear in the invariance property.

**§ 1. Projective invariance.** A Gaussian system $\{B(t\,;\omega)\,;t\in\boldsymbol{R}\}$ is called a Brownian motion if it satisfies

($\mathscr{B}$1) $B(0)\equiv 0$,

($\mathscr{B}$2) $B(t)-B(s)\overset{\mathcal{L}}{=}N(0,|t-s|)$, the Gaussian law of mean 0 and variance $|t-s|$.

To fix the idea, take a continuous version

($\mathscr{B}$3) $B(t\,;\omega)$ is continuous in $t$ including $t=\infty$ for any $\omega$, that is
$$\lim_{|t|\to\infty}\frac{1}{t}B(t)=0.$$

It is easy to show that the processes $B_{1,s}(t)$, $B_{2,u}(t)$ and $B_3(t)$ below are Brownian motions in the above sense;

($\mathscr{T}$1) $B_{1,s}(t)\equiv B(t+s)-B(s)$, $s\in\boldsymbol{R}$,

($\mathscr{T}$2) $B_{2,u}(t)\equiv e^{-u/2}B(e^u t)$, $u\in\boldsymbol{R}$,

($\mathscr{T}$3) $B_3(t)\equiv tB\left(\dfrac{-1}{t}\right)$.

It is natural to ask what group is generated by the transforms ($\mathscr{T}$1)–($\mathscr{T}$3) acting on $B(t\,;\omega)$.

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