105. *Kato's Inequality and Essential Selfadjointness for the Weyl Quantized Relativistic Hamiltonian*†)

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1. Introduction. For the nonrelativistic quantum Hamiltonian of a spinless particle of mass m, i.e. the nonrelativistic Schrödinger operator $(1/2m)(-i\partial-A(x))^2$, with magnetic fields, Kato [3] discovered a distributional inequality, which is now called *Kato's inequality*, to attack the problem of essential selfadjointness. The aim of this note is to establish an analogous distributional inequality for the Weyl quantized relativistic Hamiltonian H_A^m with magnetic fields to show the essential selfadjointness of the general Weyl quantized relativistic Hamiltonian

$$H^m = H_A^m + \Phi, \tag{1.1}$$

which corresponds to the classical relativistic Hamiltonian (e.g. [4])

$$h^m(p, x) = h_A^m(p, x) + \Phi(x) \equiv \sqrt{(p-A(x))^2+m^2} + \Phi(x), \quad p \in \boldsymbol{R}^d, \quad x \in \boldsymbol{R}^d. \tag{1.2}$$

Here m is a nonnegative constant. The vector and scalar potentials $A(x)$ $\Phi(x)$ are respectively $\boldsymbol{R}^d$-valued and $\boldsymbol{R}$-valued measurable functions in $\boldsymbol{R}^d$. It is assumed that they satisfy:

$$A(x) \quad \textit{and} \quad \int_{0<|y|<1} |A(x-y/2)-A(x)||y|^{-d}dy \quad \textit{are locally bounded,} \tag{1.3}$$

and

$$\Phi(x) \quad \textit{is in} \quad L^2_{\mathrm{loc}}(\boldsymbol{R}^d) \quad \textit{with} \quad \Phi(x) \geq 0 \quad \textit{a.e.} \tag{1.4}$$

For instance, (1.3) is fulfilled by a locally Hölder-continuous $A(x)$.

2. Statement of results. We begin with defining the Weyl quantized relativistic Hamiltonian H_A^m with magnetic fields when $A(x)$ satisfies (1.3). If $A(x)$ is sufficiently smooth and for instance, satisfies

$$|\partial^\alpha A(x)| \leq C_\alpha, \qquad x \in \boldsymbol{R}^d, \ 1 \leq |\alpha| \leq N, \tag{2.1}$$

for N sufficiently large, with a constant C_α, then it may be defined as the Weyl pseudo-differential operator $H_A^{m,w}$:

$$(H_A^{m,w}u)(x) = (2\pi)^{-d} \iint_{\boldsymbol{R}^d \times \boldsymbol{R}^d} e^{i(x-y)p} h_A^m\left(p, \frac{x+y}{2}\right) u(y)dydp, \quad u \in \mathcal{S}(\boldsymbol{R}^d). \tag{2.2}$$

The integral on the right is an oscillatory integral. Note the condition (2.1) allows the case of constant magnetic fields. The definition of H_A^m for the general $A(x)$ satisfying (1.3) is based on the *Lévy-Khinchin formula* for the conditionally negative definite function $\sqrt{p^2+m^2}$:

$$\sqrt{p^2+m^2} = m - \int_{|y|>0} [e^{ipy} - 1 - ipyI_{\{|y|<1\}}] n^m(dy). \tag{2.3}$$

Here $I_{\{|y|<1\}}$ is the indicator function of the set $\{|y|<1\}$, and $n^m(dy)$ is the *Lévy*

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