103. Ultra-hyperbolic Approach to some Multi-dimensional Inverse Problems

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1. Introduction. Our aim is to extend our previous work [2] and establish some uniqueness results for spectral and evolutional inverse problems of multi-dimensional space variables. Thus, let $\Omega\subset\boldsymbol{R}^n$ be a bounded domain with smooth boundary $\partial\Omega$ and $Pu=\nabla\cdot(a\nabla u)+cu$ be a second order formally self-adjoint uniformly elliptic differential operator with smooth coefficients $a=(a_{ij}(x))$ and $c=c(x)$ on $\bar{\Omega}$. We consider the parabolic initial boundary value problem

$$(1)\qquad \frac{\partial u}{\partial t}=Pu\ (\text{in } \Omega\times(0,\infty)),\quad u|_{t=0}=0,\quad \left.\frac{\partial u}{\partial\nu_P}\right|_{\partial\Omega}=F(\xi,t),$$

where $\partial/\partial\nu_P=\sum_{i,j}\nu_i a_{ij}(x)(\partial/\partial x_j)$, $\nu=(\nu_{ij})$ being the outer unit normal vector on $\partial\Omega$. Our concern is to determine the coefficients $a=(a_{ij})$ and c through the boundary input $F=f(\xi,t)$ and output $u=u(\xi,t)$ $(\xi\in\Gamma, 0<t<T)$, where $T>0$ and $\Gamma\subset\partial\Omega$ with $|\Gamma|>0$. Hence let Q be a similar elliptic operator and take the equation

$$(2)\qquad \frac{\partial v}{\partial t}=Qv\ (\text{in } \Omega\times(0,\infty)),\quad v|_{t=0}=0,\quad \left.\frac{\partial v}{\partial\nu_Q}\right|_{\partial\Omega}=F(\xi,t).$$

Then, our uniqueness question is formulated as follows: Does

$$(3)\qquad v(\xi,t)=u(\xi,t)\qquad (\xi\in\Gamma, 0<t<T)$$

imply $Q=P$?

2. Reduction to spectral problems. Let P_N and Q_N be the realizations in $X=L^2(\Omega)$ of the differential operators P and Q under the Neumann boundary conditions $\partial/\partial\nu_P\cdot=\partial/\partial\nu_Q\cdot=0$, respectively. The eigenvalues and eigenfunctions of $-P_N$ and $-Q_N$ are denoted by $\{\lambda_j\}$, $\{\mu_j\}$ $(-\infty<\lambda_1<\lambda_2\leqq\cdots\to+\infty,\ -\infty<\mu_1<\mu_2\leqq\cdots\to+\infty)$ and $\{\varphi_j\}$, $\{\psi_j\}$ $(\|\varphi_j\|_{L^2(\Omega)}=\|\psi_j\|_{L^2(\Omega)}=1)$, respectively. Then, supposing $F(\xi,t)=h(t)f(\xi)$ with $h\not\equiv 0$, we can deduce (e.g. [2]) from (3) that

$$(4)\qquad r(\xi,t)=s(\xi,t)\qquad (\xi\in\Gamma, 0<t<\infty),$$

where $r(x,t)=\sum_j e^{-t\lambda_j}\varphi_j(x)\int_{\partial\Omega}\varphi_j(\xi)f(\xi)d\sigma_\xi$ and $s(x,t)=\sum_j e^{-\mu_j t}\psi_j(x)\int_{\partial\Omega}x_j(\xi)\cdot f(\xi)d\sigma_\xi$. Taking $F(\xi,t)=F_l(\xi,t)=h_l(t)f_l(\xi)$ with $h_l\not\equiv 0$ for $l\in S$, we suppose the following condition, where $J_\lambda=\{j\,|\,\lambda_j=\lambda\}$ and $L_\lambda=\{j\,|\,\mu_j=\lambda\}$ for $\lambda\in\boldsymbol{R}$:

(5) The matrices $(\alpha_{jl})_{j\in J_\lambda, l\in S}$ and $(\beta_{jl})_{j\in L_\lambda, l\in S}$ are both of full-rank when $J_\lambda\neq\phi$ or $L_\lambda\neq\phi$, where $\alpha_{jl}=\int_{\partial\Omega}\varphi_j(\xi)f_l(\xi)d\sigma_\xi$ and $\beta_{jl}=\int_{\partial\Omega}\psi_j(\xi)f_l(\xi)d\sigma_\xi$.