92. *Note on Heinz's Inequality*

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The operator monotone functions are completely characterized by K. Löwner. But the proof is by no means short or elementary. For instance, it is not at all obvious that $f(t)=t^{1/2}$ is operator monotone. And in fact it was discovered by E. Heinz in 1951 that $f(t)=t^{\nu}$ was operator monotone for $\nu\varepsilon[0,1]$. In the following year T. Katô gave a shorter proof of the another Heinz's inequality.

In this note, it will be proved that Löwner's special case, Heinz's inequality, Heinz-Katô type inequality and the recent Chan-Kwong's result are all equivalent.

We use capital letters $A, B, \cdots$ to denote the bounded linear operators on the Hilbert space $\mathcal{H}$.

Theorem. *The following results* (i)–(iv) *are equivalent.*

(i) (K. Löwner) *If* $A\geqq B\geqq 0$, *then* $A^{1/2}\geqq B^{1/2}$.

(ii) (N. N. Chan–M. K. Kwong) *If* $A\geqq B\geqq 0$, $C\geqq D\geqq 0$, $AC=CA$ *and* $BD=DB$, *then* $A^{1/2}C^{1/2}\geqq B^{1/2}D^{1/2}$.

(iii) (E. Heinz) *If* $A\geqq B\geqq 0$, *then* $A^{\nu}\geqq B^{\nu}$ *for all* $\nu\varepsilon[0,1]$.

(iv) (E. Heinz–T. Katô) *If* $A\geqq 0$, $B\geqq 0$, $\|Qx\|\leqq\|Ax\|$, $\|Q^*y\|\leqq\|By\|$ *for all* $x, y\varepsilon\mathcal{H}$, *then* $|\langle Qx, y\rangle|\leqq\|A^{\nu}x\|\|B^{1-\nu}y\|$ *for all* $\nu\varepsilon[0,1]$.

To prove Theorem we need the following Lemmas.

Lemma 1. *If* $A\geqq B>0$, *then* $A^{-1}\leqq B^{-1}$.

Proof. If $A\geqq B>0$, then $B^{-1/2}AB^{-1/2}\geqq I$ and $B^{1/2}A^{-1}B^{1/2}\leqq I$ and hence $A^{-1}\leqq B^{-1}$.

Lemma 2. *If* (i) *of Theorem is fulfilled and if* $E\geqq F>0$, $X\geqq 0$, $Y\geqq 0$ *and* $XFX\geqq YEY$, *then* $X\geqq Y$.

Proof. Since $XEX\geqq XFX\geqq YEY$ by the assumptions, $E^{1/2}XEXE^{1/2}\geqq E^{1/2}YEYE^{1/2}$ and $E^{1/2}XE^{1/2}\geqq E^{1/2}YE^{1/2}$ by (i) and hence $X\geqq Y$.

Proof of Theorem. (i) *implies* (ii); For any $\varepsilon>0$, let $A_{\varepsilon}=A+\varepsilon I$, then $A_{\varepsilon}\geqq B_{\varepsilon}\geqq\varepsilon I>0$ and $B_{\varepsilon}^{-1}\geqq A_{\varepsilon}^{-1}>0$ by Lemma 1. Let $X=(A_{\varepsilon}C)^{1/2}$ and $Y=(B_{\varepsilon}D)^{1/2}$, then $X\geqq 0$, $Y\geqq 0$ and $XA_{\varepsilon}^{-1}X=(A_{\varepsilon}C)^{1/2}A_{\varepsilon}^{-1}(A_{\varepsilon}C)^{1/2}=C\geqq D=YB_{\varepsilon}^{-1}Y$ and hence $X\geqq Y$ by Lemma 2. This implies that $A^{1/2}C^{1/2}=B^{1/2}D^{1/2}$.

(ii) *implies* (iii); If $A^{\alpha}\geqq B^{\alpha}\geqq 0$, $A^{\beta}\geqq B^{\beta}\geqq 0$ for $\alpha,\beta\varepsilon[0,1]$, then $A^{(\alpha+\beta)/2}\geqq B^{(\alpha+\beta)/2}$ by (ii) and hence $A^{\nu}\geqq B^{\nu}$ for all $\nu\varepsilon[0,1]$.

(iii) *implies* (iv); Let $Q=V|Q|=|Q^*|V$ be the polar decomposition of Q, then, for any $x, y\varepsilon\mathcal{H}$, $\||Q|x\|=\|Qx\|\leqq\|Ax\|$, $\||Q^*|y\|=\|Q^*y\|\leqq\|By\|$ and $\||Q|^{\nu}x\|\leqq\|A^{\nu}x\|$, $\||Q^*|^{1-\nu}y\|\leqq\|B^{1-\nu}y\|$ for all $\nu\varepsilon[0,1]$ by (iii) and hence $|\langle Qx, y\rangle|$