86. On the Group of Units of an Abelian Extension of an Algebraic Number Field

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Let K be a finite extension of the rational number field $\boldsymbol{Q}$ and L a finite abelian extension of K. For a subextension M of L/K, we denote by E_M (resp. W_M) the group of units of M (resp. the group of roots of unity in M) and define $E_{M/K}=\{\varepsilon\in E_M \mid N_{M/F}\varepsilon\in W_F$ for all subextensions $F\neq M$ of $M/K\}$, where $N_{M/F}$ is the norm map from M to F. The elements of $E_{M/K}$ are called *relative units of M over K*. We put $\mathcal{E}_M=E_{M/K}W_L/W_L\simeq E_{M/K}/W_M$. In this note we shall prove

Theorem. *Let $\mathcal{M}$ denote the set of cyclic subextensions of L/K.*

(i) $(E_L/W_L)^{[L:K]}\subseteqq\prod_{M\in\mathcal{M}}\mathcal{E}_M$ *and the product* $\prod$ *is direct.*

(ii) *Let r_1, r_2 be the numbers of real and complex places of K, respectively, and $\boldsymbol{Z}$ the ring of rational integers. For $M\in\mathcal{M}$, let r_1^M denote the number of real places of K which are unramified in M and let $\mathfrak{O}_M$ denote the ring of integers of the $[M:K]$-th cyclotomic field. Then $\mathcal{E}_M$ is an $\mathfrak{O}_M$-module. Moreover,*

$$\mathcal{E}_M\simeq\begin{cases}\boldsymbol{Z}^{r_1+r_2-1} & \text{if } M=K,\\ 0 & \text{if } M\neq K \text{ and } r_1^M+r_2=0,\\ \mathfrak{O}_M^{r_1^M+r_2-1}\oplus\mathfrak{A}_M & \text{if } M\neq K \text{ and } r_1^M+r_2>0,\end{cases}$$

where $\mathfrak{A}_M$ is a non-zero ideal of $\mathfrak{O}_M$.

This theorem has been proved in [3] and [2] if $K=\boldsymbol{Q}$, in [5] and [4] if K is an imaginary quadratic field.

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§1. Preliminaries. Let G be an abelian group of finite order n. Let $\boldsymbol{Q}[G]$ (resp. $\boldsymbol{Z}[G]$) denote the group ring of G over $\boldsymbol{Q}$ (resp. $\boldsymbol{Z}$). Let Λ denote the set of $\boldsymbol{Q}$-irreducible characters of G. For $\lambda\in\Lambda$, we denote $G_\lambda=\{\sigma\in G\mid \lambda(\sigma)=\lambda(1)\}$, $n_\lambda=[G:G_\lambda]$ and $\Lambda_\lambda=\{\mu\in\Lambda\mid G_\lambda\subseteqq G_\mu\}$. We define

$$e_\lambda=\frac{1}{n}\sum_{\sigma\in G}\lambda(\sigma^{-1})\sigma\in\frac{1}{n}\boldsymbol{Z}[G]\subseteqq\boldsymbol{Q}[G] \quad\text{and}\quad s_\lambda=\sum_{\sigma\in G_\lambda}\sigma\in\boldsymbol{Z}[G].$$

It is easy to see that $e_\lambda^2=e_\lambda$, $e_\lambda e_\mu=0$ $(\lambda\neq\mu)$, $\sum_{\lambda\in\Lambda}e_\lambda=1$ and

$$s_\lambda=\frac{n}{n_\lambda}\sum_{\mu\in\Lambda_\lambda}e_\mu. \tag{1}$$

Let A be a G-module. Let $\bar{A}=A/TA$, where TA is the $\boldsymbol{Z}$-torsion part of A, and let $l: A\to\bar{A}$ denote the canonical surjective G-homomorphism. We note that $\bar{A}$ can be embedded into the $\boldsymbol{Q}[G]$-module $A_{\boldsymbol{Q}}=A\otimes_{\boldsymbol{Z}}\boldsymbol{Q}$ and that $A_{\boldsymbol{Q}}=\oplus_{\lambda\in\Lambda}e_\lambda A_{\boldsymbol{Q}}$. For $\lambda\in\Lambda$, we denote $A^\lambda=\{a\in A\mid \sigma a=a$ for all $\sigma\in G_\lambda\}$; then for $a\in A^\lambda$ we have