# 81. On Complexes in a Finite Abelian Group. II

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This is continued from [0].

*Proof of Theorem* 2. Let $a \in K$ and put $K'=K-a$. Then $K+K=K \circ K$ means $K'+K'=K' \circ K'$, $|K'|=k$, $|K' \circ K'|=m$. We have $0 \in K'$, and "$K+K$ is a coset of a subgroup of $G$" means "$K'+K'$ is a subgroup of $G$". So rewriting $K$ for $K'$, Theorem 2 can be reformulated as follows.

**Theorem 2′.** *Let* $0 \in K$ *and suppose* $K+K=K \circ K$. *If* $2m<3k$, *then* $K+K$ *is a subgroup of* $G$.

The proof of this theorem depends on the following

**Theorem of Kneser** ([1], see Mann [2, p. 6]). *For any complexes* $A$, $B$ *of* $G$, *there exists a subgroup* $H$ *of* $G$ *such that* $A+B=A+B+H$ *and* $|A+B| \geq |A+H|+|B+H|-|H|$.

For $A=B=K$, we obtain a subgroup $H$ such that $K+K=K+K+H$ and $|K+K| \geq 2|K+H|-|H|$. If $2m<3k$, we have $m=|K+K|<(3/2)k \leq (3/2)|K+H|$, and so $2|H|>|K+H|$. As $K+K=K+K+H$, we have $K+K \supset H$. If $(K+K)\setminus H \neq \emptyset$, there should be $x, y \in K$ such that $x+y \notin H$. Then $x$ or $y \notin H$. Suppose $x \notin H$. Then $K+H \supset (x+H) \cup H$ and $|K+H| \geq 2|H|$. Thus $K+K=H$.

**Remark.** If $G=Z/pZ$, $p$ being a prime, then $K+K=G$ or $|K+K| \geq 2|K|-1$. This follows from the theorem of Kneser or from Cauchy-Davenport's theorem (see Mann [2, p. 3]).

Let $G$ be any other abelian group than $Z|pZ$ and $H$ a non-trivial subgroup of $G$ (i.e. $H \neq \{0\}$, $H \neq G$). Put $K=H \cup (x+H)$, $2x \notin H$. Then $|K+K| =(3/2)|K|$, so that 3/2 in (ii) can not be replaced by a smaller number.

Since $K \circ K \neq K+K$, in order to prove Theorem 3 we may suppose $K$ satisfies (0). Moreover we may prove Theorem 3 for $K$ with the following maximality property: there is no $s \in G\setminus K$ such that

$$(*) \qquad |(K \cup \{s\}) \circ (K \cup \{s\})| \leq |K \circ K|+1.$$

In fact, if there exists $s \in G\setminus K$ which satisfies $(*)$, then $K'=K \cup \{s\}$ satisfies (0) and if Theorem 3 is proved with respect to $K'$, then the inequality also holds true for $K$.

**Lemma 4.** *If* $|G|$ *is odd,* $K$ *satisfies* (0) *and there is no* $s \in G\setminus K$ *which satisfies* $(*)$, *then* $|K^w| \leq m-k+3$ *for every* $w \in (K \circ K)\setminus K$.

*Proof.* Suppose $|K^w| \geq m-k+4$ for some $w \in (K \circ K)\setminus K$. Put $K_{(x)}=\{y \in K\setminus\{x, 0\} \mid x+y \in K\}$ for $x$ $(\neq 0) \in K$, then $K_{(x)} \to K \cap K_x$, $y \to x+y$ is a

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