59. *Classification of Normal Congruence Subgroups of $G(\sqrt{q})$.* II

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This is continued from [0].

4. Here we treat the case of general level L. Any L can be written uniquely as $L=\prod L_p$ ($L_p \in \boldsymbol{L}$), where p runs over all primes and L_p is a power of p. Then we have the canonical isomorphism $H_q(L) \cong \prod_{p|L} H_q(L_p)$, where $p|L$ means $L_p \neq 1$. We regard $H_q(L_p)$ as a subgroup of $H_q(L)$ by this isomorphism. A set $\{N_1, \cdots, N_k\}$ of normal σ-subgroups of level L of $H_q(L)$ is called a *Z-complete set* of $H_q(L)$ if any normal σ-subgroup N of level L of $H_q(L)$ can be expressed as $N=N_iZ$ ($1 \leqq i \leqq k$) by a σ-subgroup Z of $Z_q(L)$, where $Z_q(L)$ denotes the center of $H_q(L)$. Let $\mathfrak{S}=\{N_1, \cdots, N_k\}$ be a set of normal subgroups of $H_q(L)$ and K be a normal subgroup of $H_q(L)$. Then $\mathfrak{S}K$ denotes the set $\{N_1K, \cdots, N_kK\}$.

In order to define some normal σ-subgroups of level L of $H_q(L)$, we shall use the notation $[F, n; z]$ defined as follows. Let G_1 and G_2 be any two groups, and set $G=G_1 \times G_2$. Let (F, n) be a pair of a normal subgroup F of G_1 and an element n of G_1. Let z be an element of the center of G_2. Then we set $[F, n; z]=F \times \langle z^2 \rangle \cup nF \times z\langle z^2 \rangle$.

For any integer $k \in \boldsymbol{N}$, put $L_k^* = \prod_{p \nmid k} L_p$. Suppose now that $q \neq 2$. When $L_2^* \neq 1$, the subset Z_q^* of $Z_q(L_2^*)$ is defined by $Z_q^* = \{z \in Z_q(q^{1/2}) \prod_{p|L_{2q}^*} \{\pm I_p\} \mid \mathrm{ord}(z)=\text{even}\}$ (if $L_q=q^{1/2}$) or $\{z \in \prod_{p|L_2^*}\{\pm I_p\} \mid z \neq I\}$ (if $L_q \neq q^{1/2}$). Let us define the set $\mathfrak{S}_q(L)$ of subgroups of $H_q(L)$ by $\mathfrak{S}_q(L)=\{1\}$ (if $L_2=1$), $\{1, Q_1, [Q_1, B; z]\ (z \in Z_q^*)\}$ (if $L_2=2$), $\{1, E_2, Q_2\}$ (if $L_2=2^2$), $\{1, E_3\}$ (if $L_2=2^3$), $\{1, E_m, G_m^+, G_m^-, [F_m^+, X; z]\ (z \in Z_q^*), [F_m^+, -X; z]\ (z \in Z_q^*)\}$ ($X=\phi^{-1}(B_{m-1}C_{m-1} \cdot D_{m-4})$) (if $L_2=2^m$, $m \geqq 4$), where the groups of type $[F, n; z]$ are defined with respect to the decomposition $H_q(L)=H_q(2^m) \times H_q(L_2^*)$.

Theorem 4. *Assume that $q \neq 2$. Let L be any element of $\boldsymbol{L}$. Let $\mathfrak{S}_q(L)$ be the set defined above. Then a Z-complete set of $H_q(L)$ is given by the union of $\mathfrak{S}_q(L)$, $\mathfrak{S}_q(L)M$, $\mathfrak{S}_5(L)R_k^{(5)}$, $\mathfrak{S}_5(L)R_k^{(5)}M$, $\mathfrak{S}_5(L)S_k^{(5)}$, $\mathfrak{S}_5(L)S_k^{(5)}M$, where the sets multiplied by $R_k^{(5)}$ or $S_k^{(5)}$ appear only when $q=5$ and $L_5=5^k$ ($k \in \boldsymbol{N}$), and the sets multiplied by M appear only when $q \neq 3$ and $L_3=3$.*

Suppose now that $q=2$. When $L_2^* \neq 1$, set $Z_2^*=\{z \in \prod_{p|L_2^*}\{\pm I_p\} \mid z \neq I\}$. Let us define the set $\mathfrak{S}_2(L)$ of subgroups of $H_2(L)$ by $\mathfrak{S}_2(L)=\{1\}$ (if $L_2=2^{m-1/2}$ ($m \geqq 2$), 1, 2), $\{1, R_2, S_2, [\pm E_2^+, BC; z], [\pm E_2^+, BC^{-1}; z]\}$ (if $L_2=2^2$), $\{1, L_3^+, L_3^-, M_3^+, M_3^-, P_3, Q_3, S_3^+, S_3^-, [H_3^+, B_1C_1; z], [H_3^+, -B_1C_1; z], [H_3^+, B_1C_1^{-1}; z], [H_3^+, -B_1C_1^{-1}; z], [\pm \mathrm{L}_3^+, BC^{-1}; z], [\pm L_3^+, BC^{-1}D; z], [E_2^{3+}, BC; z], [E_2^{3+}, -BC; z]\}$ (if $L_2=2^3$), $\{1, L_m^+, L_m^-, M_m^+, M_m^-, N_m^+, N_m^-, O_m^+, O_m^-, [H_m^+, L; z], [H_m^+,$