50. Zeros of $L(s,\chi)$ in Short Segments on the Critical Line

By Yoichi MOTOHASHI
Department of Mathematics, College of Science and Technology, Nihon University

(Communicated by Kunihiko KODAIRA, M. J. A., May 12, 1988)

1. Let $L(s,\chi)$ be the Dirichlet L-function with χ primitive $(\mathrm{mod}\, k)$, $k>1$. Let $N_0(T,\chi)$ be the number of zeros of $L(s,\chi)$ on the segment $s=1/2+it$, $0\leqq t\leqq T$. The purpose of the present note is to give a brief proof of

Theorem. *Let $T\geqq k^{(1/2)+6\varepsilon}$, $U\geqq (kT)^{(1/3)+2\varepsilon}$ with small $\varepsilon>0$. Then we have*

$$N_0(T+U,\chi)-N_0(T,\chi)\gg_\varepsilon U\log T.$$

This should be compared with Karatsuba [2], and we stress that a minor modification of our argument can yield a slight improvement upon his result. There are two important ingredients in our argument: One is Atkinson's method [1], and the other is Weil's result [6] on character sums. More specifically, we have combined Selberg's ideas [5] with ours [3]–[4].

2. Here we outline our proof of the theorem. The details will be published elsewhere.

Let $L(s,\chi)=\psi(s,\chi)L(1-s,\bar\chi)$ be the functional equation for $L(s,\chi)$, and put $X(t,\chi)=\psi^{-1/2}(1/2+it,\chi)L(1/2+it,\chi)$ which is real for real t. Also, as in [5], let $\alpha(\nu)$ be the coefficient in the Dirichlet series expansion for $\zeta(s)^{-1/2}$, and let $\beta(\nu)=\alpha(\nu)(\log \xi/\nu)/\log\xi$ with ξ to be determined later. We put

$$\eta(t,\chi)=\sum_{\nu<\xi}\chi(\nu)\beta(\nu)\nu^{-(1/2)-it}.$$

And we consider the estimation of

$$I=\int_{-U\log T}^{U\log T}\left|\int_0^H X(T+t+u,\chi)|\eta(T+t+u,\chi)|^2du\right|^2 e^{-(t/U)^2}dt,$$

$$J=\int_{-U\log T}^{U\log T}\left|\int_0^H L\Big(\frac12+i(T+t+u),\chi\Big)\eta^2(T+t+u,\chi)du-H\right|^2 e^{-(t/U)^2}dt,$$

where $H\ll 1$, $(kT)^{(1/3)+2\varepsilon}\leqq U\leqq T^{1-\varepsilon}$.

Invoking the result of [4] we have, as a first step,

$$\begin{aligned} I\ll U\xi^2T^{-\varepsilon}+\int_0^H\!\!\int_0^H\Big(\frac{kT}{2\pi}\Big)^{(i/2)(u-v)}\sum_{\nu<\xi}\frac{\chi(\nu_1\nu_2)\bar\chi(\nu_3\nu_4)}{(\nu_1\nu_2\nu_3\nu_4)^{1/2}}\Big(\frac{\nu_3}{\nu_1}\Big)^{iu}\Big(\frac{\nu_4}{\nu_2}\Big)^{iv}\beta(\nu_1)\beta(\nu_2)\beta(\nu_3)\beta(\nu_4) \\ \times\int_{-U\log T}^{U\log T}e^{-(t/U)^2}L\Big(\frac12+i(T+t+u),\chi\Big)L\Big(\frac12-i(T+t+v),\bar\chi\Big) \\ \times\Big(\frac{\nu_3\nu_4}{\nu_1\nu_2}\Big)^{i(T+t)}dt\,du\,dv. \end{aligned}\tag{1}$$

Then we apply a modified version of Atkinson's splitting argument to this product of values of L-functions. For this sake let a, b be two positive integers such that $(a,b)=1$ and $(ab,k)=1$. And we write, for $\mathrm{Re}\,(z)>1$, $\mathrm{Re}\,(w)>1$,