47. *Azumaya Algebras Split by Real Closure*[†]

By Teruo KANZAKI*) and Yutaka WATANABE**)

(Communicated by Shokichi IYANAGA, M. J. A., May 12, 1988)

1. Introduction. Let K be a commutative ring with identity element. For a (local) signature $\sigma: K \to \mathrm{GF}(3)=\{0, \pm 1\}$, (which satisfies $\sigma(-1)=-1$, for any $a, b \in K$ $\sigma(ab)=\sigma(a)\sigma(b)$, and $\sigma(a)=0$ or $\sigma(a)=\sigma(b)$ imply $\sigma(a+b)=\sigma(b)$ cf. [4]), $P_\sigma=\{x \in K \mid \sigma(x)=0 \text{ or } 1\}$ satisfies the following conditions; $P_\sigma+P_\sigma \subseteq P_\sigma$, $P_\sigma \cdot P_\sigma \subseteq P_\sigma$, $P_\sigma \cup (-P_\sigma)=K$, and $\mathfrak{p}_\sigma=P_\sigma \cap (-P_\sigma)$ is a prime ideal of K. Then P_σ is an ordering in the meaning of [6]. Conversely, an ordering P of K defines a signature $\sigma_P: K \to \mathrm{GF}(3)$; $\sigma_P(x)=0$ if $x \in P \cap (-P)$, $\sigma_P(x)=1$ if $x \in P$ and $x \notin -P$, and $\sigma_P(x)=-1$ if $x \in -P$ and $x \notin P$. Therefore, we can identify σ and P_σ, (or P and σ_P). By $\mathrm{Sig}(K)$, we denote the set $\{\sigma: K \to \mathrm{GF}(3) \mid \text{signature on } K\}$ $(=\{P \mid \text{ordering on } K\})$. Let P_0 be an ordering on K. For the prime ideal $\mathfrak{p}_0=P_0 \cap (-P_0)$ of K, $(\bar{K}_0, \bar{P}_0)$ denotes the totally ordered quotient field of the totally ordered domain $(K/\mathfrak{p}_0, P_0/\mathfrak{p}_0)$, and R_0 the real closure of the totally ordered field $(\bar{K}_0, \bar{P}_0)$. Let A be a K-algebra with identity element such that A is a finitely generated projective K-module. Then, there are elements $a_1, a_2, \cdots, a_n \in A$ and $\psi_1, \psi_2, \cdots, \psi_n \in \mathrm{Hom}_K(A, K)$ such that $a=\sum_{i=1}^n \psi_i(a)a_i$ for all $a \in A$. The trace map $t_r: A \to K; a \rightsquigarrow \sum_{i=1}^n \psi_i(aa_i)$ defines a quadratic K-module (A, ρ) by $\rho(a)=\mathrm{tr}(a^2)$ for $a \in A$. If $L \supset K$ is a commutative Galois extension with a finite Galois group G, then $\mathrm{tr}(a)=t_G(a) := \sum_{\sigma \in G} \sigma(a)$ holds for all $a \in A$ (cf. [2]). Let A be an Azumaya K-algebra. We shall say A to be P_0-split, if $A \otimes_K R_0$ is a matrix ring over R_0. Furthermore, we shall say that A is *real split*, if A is P-split for all $P \in \mathrm{Sig}(K)$. By $B(K, P_0)$ and $B^r(K)$, we denote the subgroups $\{[A] \in B(K) \mid A: P_0\text{-split}\}$ and $\{[A] \in B(K) \mid A: \text{real split}\}$ of the Brauer group $B(K)$ of K, respectively. Then, $B^r(K)=\bigcap_{P \in \mathrm{Sig}(K)} B(K, P)$. Let $L \supseteq K$ be a commutative ring extension with common identity element. Then we put $\mathrm{Sig}_{P_0}(L/K) := \{P \in \mathrm{Sig}(L) \mid P \cap K=P_0\}$, and $Q(K) := \bigcap_{P \in \mathrm{Sig}(K)} P$. $Q(K \mid L)$ denotes the intersection of all P in $\mathrm{Sig}(K)$ such that $\mathrm{Sig}_P(L/K)=\varnothing$. A quadratic K-module (M, q) is said to be *positive semi-definite*, if $q(x)$ belongs to $Q(K)$ for all $x \in M$. In this paper, we prove the following theorem:

Theorem. *Let $L \supset K$ be a Galois extension of commutative rings with a finite Galois group G in the meaning of* [2]. *Then, the following assertions hold*:

1) *If the quadratic K-module (L, ρ) is positive semi-definite, then $B(L/K)(:=\{[A] \in B(K) \mid A \otimes_K L \sim L$: A is split by $L\})$ is included in $B^r(K)$.*

---

*) Department of Mathematics, Kinki University.

**) Department of Mathematics, Osaka Women's University.

†) In memory of Professor Akira HATTORI.